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Reports of the Academy of Sciences of the USSR

1965

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Abstract

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Reports of the Academy of Sciences of the USSR
1965. Volume 165, No. 4

MATHEMATICS

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LINEAR TRANSFORMATIONS OF DIRECTIONALLY DIVERGENT SEQUENCES

(Presented by Academician A. N. Kolmogorov, 15 IV 1965)

In the present note we consider unbounded complex sequences tending to infinity in a certain direction along a half-strip or along an angle, and their linear transformations by complex regular matrices (T-matrices) with finite rows.

§ 1. Let us introduce some definitions. We shall call a **half-strip** in the complex plane a closed domain bounded by some segment and by two parallel rays issuing in the same direction from the endpoints of this segment, perpendicular to it. A half-strip is completely determined by three parameters: the length of the segment H, $H \geq 0$ (the **width of the half-strip**), the point z at which the midpoint of this segment is located (the **initial point** of the half-strip), and the angle α, $0 \leq \alpha < 2\pi$, between the positive direction of the real axis and the direction of the rays bounding the half-strip (the **direction of the half-strip**). A half-strip with initial point z, direction α, and width H will be denoted by $\Pi(z, \alpha, H)$.

We shall say that a sequence of complex numbers $\{s_n\}$ **tends to infinity along the half-strip** $\Pi(z, \alpha, H)$, and denote this by

$$s_n \to \infty \Pi(z, \alpha, H),$$

if: 1) $s_n \in \Pi(z, \alpha, H)$ for $n > N$ (the number N generally depends on $\{s_n\}, z, \alpha, H$); 2) $\lim_{n\to\infty} |s_n| = \infty$. The half-strip $\Pi(z, \alpha, H)$ itself in this case will be called the **half-strip of divergence** of the sequence $\{s_n\}$ to ∞.

The following question is posed: to find conditions on the elements of a T-matrix (a_{mn}) with finite rows such that every sequence $\{s_n\}$ tending to infinity along some half-strip is transformed by this matrix into a sequence $\{\sigma_m\}$,

$$\sigma_m = \sum_{n=1}^{\infty} a_{mn} s_n,$$

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also tending to ∞ along some half-strip. Since a half-strip is determined by the three parameters z, α, and H, various cases are possible here, depending on which of the parameters are to be changed or, conversely, preserved under the linear transformation under consideration.

The following results give an answer to the question posed. In the first of them, conditions are established on the elements of the matrix under which the half-strip of divergence is preserved exactly.

Theorem 1. *In order that, from the divergence of a sequence* $\{s_n\}$ *to* ∞ *along any half-strip* $\Pi(z, \alpha, H)$*, there should always follow the divergence of the transformed sequence* $\{\sigma_m\}$ *to* ∞ *along the very same half-strip* $\Pi(z, \alpha, H)$*, it is necessary and sufficient that the matrix* (a_{mn}) *satisfy the following conditions:*

1) there exists an n_0 such that

$$\operatorname{Re} a_{mn} \geq 0, \qquad \operatorname{Im} a_{mn} = 0, \qquad \text{for } n > n_0, \quad m = 1, 2, \dots;$$

2) for each n one can specify an $M = M(n)$ such that

$$a_{mn} = 0 \qquad \text{for } m > M(n), \quad n = 1, 2, \dots;$$

3) there exists an m_0 such that

$$\sum_{n=1}^{\infty} a_{mn} = 1 \qquad \text{for } m > m_0.$$

If, for the half-line of divergence of the transformed sequence, one permits an ε-enlargement in comparison with the half-line of divergence of the original sequence, then the requirements on the matrix are weakened, as the following theorem shows.

Theorem 2. In order that from the divergence of a sequence $\{s_n\}$ to ∞ along any half-line $\Pi(z, \alpha, H)$ there always follow the divergence of the transformed sequence $\{\sigma_m\}$ to ∞ along the half-line $\Pi(z, \alpha, H + \varepsilon)$ for the same values of z, α, H and for any $\varepsilon > 0$, it is necessary and sufficient that the matrix (a_{mn}) satisfy condition 1) of Theorem 1.

The following theorems show that there do not exist T-matrices with finite rows which transform every sequence diverging to ∞ along some half-line into a sequence diverging to ∞ along a narrower or essentially different half-line (i.e., along a half-line which is not an ε-enlargement of the half-line of divergence of the original sequence).

Theorem 3. There does not exist a T-matrix (a_{mn}) with finite rows which would transform every sequence $\{s_n\}$ diverging to ∞ along some half-line $\Pi(z, \alpha, H)$ into a sequence $\{\sigma_m\}$ diverging to ∞ along a narrower half-line (i.e.,

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along the half-line $\Pi(z, \alpha, H')$ for the same values of z and α and with $H' < H$, where dependence of H' on the sequence $\{s_n\}$ is allowed).

Theorem 4. If a T-matrix (a_{mn}) with finite rows transforms every sequence $\{s_n\}$ diverging to ∞ along some half-line $\Pi(z, \alpha, H)$ into a sequence $\{\sigma_m\}$ also diverging to ∞ along some half-line $\Pi(z', \alpha', H')$, where dependence of z', α', and H' on $\{s_n\}$ is allowed, then it necessarily satisfies condition 1) of Theorem 1 (and hence in fact transforms every sequence $\{s_n\}$ diverging to ∞ along the half-line $\Pi(z, \alpha, H)$ into a sequence $\{\sigma_m\}$ diverging to ∞ along the half-line $\Pi(z, \alpha, H + \varepsilon)$ for any $\varepsilon > 0$).

§ 2. The following results concern the transformation of sequences diverging to infinity in an angle.

Let us introduce several definitions. By an angle in the complex plane we shall mean a closed region bounded by two rays issuing from some point z. An angle is completely determined by three parameters: the point z (the vertex of the angle), the size of the linear angle φ between the rays (the magnitude of the angle), and the angle α between the positive direction of the real axis and the bisector of the angle φ (the direction of the angle). An angle with vertex at the point z, magnitude φ, and direction α will be denoted by $\Phi(z, \alpha, \varphi)$. Throughout this note only angles smaller than π will be considered, i.e. $0 \leq \varphi < \pi$.

We shall say that a sequence of complex numbers $\{s_n\}$ diverges to infinity in the angle $\Phi(z, \alpha, \varphi)$, and denote this by

$$s_n \to \infty \, \Phi(z, \alpha, \varphi),$$

if: 1) $s_n \in \Phi(z, \alpha, \varphi)$ for $n > N$ (the number N generally depends on $\{s_n\}, z, \alpha$ and φ); 2) $\lim\limits_{n \to \infty} |s_n| = \infty$.

The angle $\Phi(z, \alpha, \varphi)$ itself will in this case be called the **angle of divergence** of the sequence $\{s_n\}$ to infinity.

As in § 1, the question is posed of finding conditions on the elements of a T-matrix (a_{mn}) with finite rows such that, under the linear transformation by means of this matrix, divergence to ∞ in the angle of the original sequence $\{s_n\}$ entails divergence in an angle of the transformed sequence $\{\sigma_m\}$. We begin with a theorem establishing conditions under which the angle of divergence is preserved.

Theorem 5. *In order that, from the divergence of the sequence $\{s_n\}$ to ∞ in any angle $\Phi(z, \alpha, \varphi)$, there should always follow the divergence of the sequence $\{\sigma_m\}$ to ∞ in the same angle, it is necessary and sufficient that the matrix (a_{mn}) satisfy conditions 1), 2), and 3) of Theorem 1.*

Denote by $\Phi_\varepsilon(z, \alpha, \varphi)$ the domain consisting of the angle $\Phi(z, \alpha, \varphi)$ and the points of the complex plane at distance not greater than ε from the sides of the angle. If we allow the transformed sequence $\{\sigma_m\}$ to diverge to ∞ in such a

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domain when $\{s_n\}$ diverges to ∞ in the angle $\Phi(z, \alpha, \varphi)$, then the restrictions on the matrix are weakened, as is seen from the following theorem.

Theorem 6. *In order that, from the divergence of the sequence $\{s_n\}$ to ∞ in any angle $\Phi(z, \alpha, \varphi)$, there should always follow the divergence of the sequence $\{\sigma_m\}$ to ∞ in the domain $\Phi_\varepsilon(z, \alpha, \varphi)$, for the same values z, α, φ and any $\varepsilon > 0$, it is necessary and sufficient that the matrix (a_{mn}) satisfy condition 1) of Theorem 1.*

The restrictions on the matrix are weakened still further if one allows an ε-enlargement of the angle of divergence; namely, the following holds.

Theorem 7. *In order that, from the divergence of the sequence $\{s_n\}$ to infinity in any angle $\Phi(z, \alpha, \varphi)$, there should always follow the divergence of the sequence $\{\sigma_m\}$ to ∞ in the angle $\Phi(z, \alpha, \varphi + \varepsilon)$, for the same values z, α, φ and any $\varepsilon > 0$ $(\varphi + \varepsilon < \pi)$, it is necessary and sufficient that the matrix (a_{mn}) satisfy the first of the inequalities in condition 1) of Theorem 1 and, in addition,*

$$\lim_{\substack{m\to\infty \\ n\to\infty}} \frac{\operatorname{Im} a_{mn}}{\operatorname{Re} a_{mn}} = 0.$$

The following theorem establishes that there exists no T-matrix that compresses the angle of divergence of every sequence.

Theorem 8. *There exists no T-matrix with finite rows which would transform every sequence $\{s_n\}$ diverging to ∞ in some angle $\Phi(z, \alpha, \varphi)$ into a sequence $\{\sigma_m\}$ diverging to ∞ in a smaller angle (i.e. in an angle $\Phi(z, \alpha, \varphi')$ for the same values of z and α and with $\varphi' < \varphi$, where dependence of φ' on $\{s_n\}$ is allowed).*

Theorem 9. *If a T-matrix (a_{mn}) with finite rows is such that every sequence $\{s_n\}$ diverging to ∞ in some angle $\Phi(z, \alpha, \varphi)$ is transformed into a sequence $\{\sigma_m\}$ diverging to ∞ also in some angle $\Phi(z', \alpha', \varphi')$ (where dependence of z', α', φ' on $\{s_n\}$ is allowed), then it necessarily satisfies the conditions of Theorem 7.*

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Received
15 IV 1965

Note: Figure translations are in progress. See original paper for figures.

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