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Abstract

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ON THE STABILITY OF STATIONARY FLOWS OF A VISCOUS INCOMPRESSIBLE FLUID

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In studying the stability of stationary flows of a viscous incompressible fluid, one usually uses the method of small oscillations. Stability or instability is judged from the location of the spectrum of the spatial operator of the equations in variations. The validity of the method, especially in establishing instability, has been questioned by a number of authors (see ([1,2])). Below a justification of this method is given.

1. Let a viscous incompressible fluid fill a three-dimensional domain Ω, and suppose that, for given external forces $\mathbf{F}$ and velocity $\mathbf{a}(x)$ on the boundary S, there exists a stationary solution of the Navier–Stokes equations $\mathbf{v}_0(x)$, $P_0(x)$ (x is a point of the domain Ω; $\mathbf{v}_0$ is the velocity, P_0 the pressure). Taking the velocity and pressure of any flow with the same $\mathbf{F}$, $\mathbf{a}$ in the form $\mathbf{v} = \mathbf{v}_0 + \mathbf{u}(x,t)$, $P = P_0 + Q(x,t)$, and putting $\mathbf{u}(x,0) = \mathbf{a}(x)$, we arrive at the problem

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{v}_0, \nabla)\mathbf{u} + (\mathbf{u}, \nabla)\mathbf{v}_0 - \nu\Delta\mathbf{u} = -\nabla Q - (\mathbf{u}, \nabla)\mathbf{u}, \tag{1}$$

$$\operatorname{div}\mathbf{u} = 0, \tag{2}$$

$$\mathbf{u}\big|_S = 0, \tag{3}$$

$$\mathbf{u}\big|_{t=0} = a(x). \tag{4}$$

Discarding the nonlinear term in (1), we arrive at the equations in variations. If solutions of the latter are sought in the form $\mathbf{u} = e^{-\lambda t}\mathbf{w}(x)$, $Q = e^{-\lambda t}q(x)$, we obtain the spectral problem

$$-\lambda\mathbf{w} + (\mathbf{v}_0, \nabla)\mathbf{w} + (\mathbf{w}, \nabla)\mathbf{v}_0 - \nu\nabla\mathbf{w} = -\nabla q, \tag{5}$$

$$\operatorname{div} \mathbf{w} = 0, \tag{6}$$

$$\mathbf{w}\big|_S = 0. \tag{7}$$

In what follows we shall assume Ω to be a bounded domain of class $C^{(2)}$. By S_p $(p > 1)$ we denote the closure, in the L_p-norm, of the set M of smooth solenoidal vectors vanishing on S. On the set of solenoidal vectors from the S. L. Sobolev space $W_p^{(2)}$ which vanish on S, define the operator A by setting

$$A\mathbf{u} = -\Pi[\nu\Delta\mathbf{u} - (\mathbf{v}_0, \nabla)\mathbf{u} - (\mathbf{u}, \nabla)\mathbf{v}_0], \tag{8}$$

where Π is the operator of orthogonal projection in L_2 of L_p onto S_p. Problems (1)–(4) and (5)–(7) can now be written in the form

$$\frac{d\mathbf{u}}{dt} + A\mathbf{u} = K\mathbf{u}, \qquad \mathbf{u}\big|_{t=0} = \mathbf{a}, \tag{9}$$

$$A\mathbf{w} = \lambda\mathbf{w}, \tag{10}$$

where $K\mathbf{u} = -\Pi(\mathbf{u}, \nabla)\mathbf{u}$. Here it is taken into account that $\Pi(\operatorname{grad} q) = 0$.

2. **Lemma 1.** *The operator* Π, *initially defined on* M, *admits an extension to a bounded operator from* L_p *to* S_p.

Lemma 2. 1) *The operator* A *in* S_p *is closed; its spectrum consists of eigenvalues* $\lambda_1, \lambda_2, \dots$; 2) *the corresponding eigenvectors and associated vectors form a complete system in* S_2; 3) *as* $k \to \infty$, $\operatorname{Re}\lambda_k \to +\infty$; 4) *there exists at most a finite number of* λ_k *with* $\operatorname{Re}\lambda_k < 0$.

We note that item 2) was established by S. G. Krein (3).

Let $\Sigma_{R,\varphi}$ be the set of points of the complex plane:

$$|\lambda| \geq R; \qquad \pi \geq |\arg\lambda| \geq \varphi > 0.$$

Lemma 3. *For any* $0 < \varphi < \pi$ *one can choose* R *so large that, uniformly in* $\lambda \in \Sigma_{R,\varphi}$, *the estimates*

$$\|(\lambda I - A)^{-1}\|_{S_p \to S_p} \leq \frac{C}{|\lambda|}, \qquad \|(\lambda I - A)^{-1}\|_{S_p \to W_p^{(2)}} \leq C \tag{11}$$

hold.

Lemma 3 makes it possible, on the basis of the results of (4, 5), to assert that $-A$ is the generating operator of a strongly continuous semigroup of operators $U(t)$ in S_p, analytic in every sector $|\arg t| < \theta$, $\theta \in (0, \pi/2)$.

In proving Lemmas 1-3 one uses: 1) energy relations for equation (10); 2) results of M. V. Keldysh (6); 3) some methods of the works (7, 8).

3. Consider the linear problem

$$\frac{d\mathbf{u}}{dt} + A\mathbf{u} = \mathbf{f}; \qquad \mathbf{u}\big|_{t=0} = \mathbf{a}. \tag{12}$$

The solution has the form

$$\mathbf{u}(t) = U(t)\mathbf{a} + \int_0^t U(t-\tau)\mathbf{f}(\tau)\,d\tau = K_1\mathbf{a} + K_2\mathbf{f}. \tag{13}$$

Lemma 4. *Let the spectrum of the operator A be contained in the half-plane* $\operatorname{Re}\lambda \geq \sigma_0$; $\mathbf{a} \in S_p$. *Then $\mathbf{u}_1(t) = K_1\mathbf{a}$ is a function continuous in t for $t \geq 0$, with values in S_p, and for any $\sigma < \sigma_0$ the inequalities*

$$\|\mathbf{u}_1(t)\|_{S_p} \leq C_\sigma e^{-\sigma t}\|\mathbf{a}\|_{S_p}, \tag{14}$$

$$\int_0^\infty \|e^{\sigma t}\mathbf{u}_1(t)\|^{5/3p}_{S^{5/3p}_{5/3p}}\,dt \leq C_\sigma \|\mathbf{a}\|^{5/3p}_{S_p}, \tag{15}$$

$$\int_0^\infty \|e^{\sigma t}\mathbf{u}_1(t)\|^r_{W_r^{(1)}(\Omega)}\,dt \leq C_\sigma\|\mathbf{a}\|^r_{S_p}, \qquad r = \frac{5p}{p+3}. \tag{16}$$

hold.

By C_σ are denoted various constants depending only on σ, p.

Lemma 5. *Let the spectrum of A be contained in the half-plane* $\operatorname{Re}\lambda \geq \sigma_0$ *and*

$$M_\sigma(\mathbf{f}) = \left[\int_0^\infty \|e^{\sigma t}\mathbf{f}(t)\|^{p_1}_{S_{p_1}}\,dt\right]^{1/p_1} < \infty \qquad (\sigma < \sigma_0).$$

Then $\mathbf{u}_2 = K_2\mathbf{f}$ is a function continuous in t with values in

$$S_p\left(p = \frac{3p_1}{5-2p_1};\ \frac{5}{2} > p_1 > 1\right)$$

and the estimates

$$\|\mathbf{u}_2(t)\|_{S_p} \leq C_\sigma e^{-\sigma t} M_\sigma(\mathbf{f}), \tag{17}$$

$$\int_0^\infty \|e^{\sigma t}\mathbf{u}_2(t)\|^{5/3p}_{S^{5/3p}_{5/3p}}\,dt \leq C_\sigma[M_\sigma(\mathbf{f})]^{5/3p}, \tag{18}$$

$$\int_0^\infty \|e^{\sigma t}\mathbf{u}_2(t)\|^r_{W_r^{(1)}(\Omega)}\,dt \leq C_\sigma[M_\sigma(\mathbf{f})]^r, \qquad r = \frac{5p_1}{5-p_1} \quad (5 > p_1 > 1). \tag{19}$$

hold.

We note that these lemmas contain some new theorems on the embedding and extension of functions defined in a cylinder. The proof of Lemmas 4-5 uses integral representations of the operators K_1, K_2, given by semigroup theory ([4,5]), and Lemma 3. It can be proved that the exponents in the left-hand sides of inequalities (14)–(19) are sharp.

4. Now problem (9) is reduced to the integral equation

$$\mathbf{u}(t) = U(t)\mathbf{a} + \int_0^t U(t-\tau)K\mathbf{u}(\tau)\,d\tau. \tag{20}$$

We shall consider solutions of (20) in the cylinders $Q_T = \Omega \times [0,T]$ $(0 < T < \infty)$ such that $\mathbf{u}(t)$ is a continuous function of t on $[0,T]$ with values in S_p, and moreover

$$\|\mathbf{u}(t)\|_{L^{5/3}p(Q_T)} < \infty, \qquad \|D_x\mathbf{u}(t)\|_{L^{\frac{5p}{p+3}}Q_T} < \infty, \qquad p \geqslant 3.$$

Definition 1. Let E, E_1 be Banach spaces. We shall say that the stationary flow $\mathbf{v}_0(x), P_0(x)$ is **positively Lyapunov stable** (E, E_1) if: 1) for any $\mathbf{a} \in E$ from some ball $\|\mathbf{a}\|_E < \delta_0$, (20) has a solution $N^t\mathbf{a} = \mathbf{u}(t) \in E_1$ for every $t > 0$, and 2) for every $\varepsilon > 0$ there is a $\delta > 0$ such that from $\|\mathbf{a}\|_E < \delta$ it follows that $\|\mathbf{u}(t)\|_E < \varepsilon$ for $t > 0$.

Definition 2. We shall say that $(\mathbf{v}_0, P_0)$ is η-stable (E', E_1) if condition 1) of Definition 1 is satisfied and, for every pair of numbers $\eta > 0$, $\varepsilon > 0$, there is a $\delta > 0$ such that from $\|\mathbf{a}\|_E < \delta$ it follows that $\|\mathbf{u}(t)\|_{E_1} < \varepsilon$ for $t > \eta$. Adding the word "asymptotically" means that $\|\mathbf{u}(t)\|_{E_1} \to 0$ as $t \to \infty$. In the language of η-stability it is convenient to describe the property of improvement, with the passage of time, of the differential properties of solutions of parabolic problems.

Theorem 1. *Let the spectrum of the operator A be located in the half-plane* $\operatorname{Re}\lambda > 0$. *Then the flow $(\mathbf{v}_0, P_0)$ is positively asymptotically Lyapunov stable (S_p, S_p) for any $p \geqslant 3$.*

Theorem 2. *Let the operator A have at least one eigenvalue in the half-plane* $\operatorname{Re}\lambda < 0$. *Then the flow $(\mathbf{v}_0, P_0)$ is unstable $(C^{(k)}, S_p)$ for $p \geqslant 3$ and any k.*

In Theorem 2, $C^{(k)}$ may be replaced by any space dense in S_p.

Theorems 1 and 2 are true in the two-dimensional case with the number 3 replaced by 2.

We give one result on η-stability.

Theorem 3. *Let $\mathbf{F}, S, \alpha$ be infinitely differentiable, and let the spectrum of A lie in the half-plane* $\operatorname{Re}\lambda > 0$. *Then the flow $(\mathbf{v}_0, P_0)$ is positively asymptotically η-stable $(S_p, C^{(k)})$ for any k and any $p \geqslant 3$.*

5. An unstable stationary flow turns out to be conditionally stable when a finite number of constraints are imposed.

Theorem 4. *Let m be the dimension of the invariant subspace E^- of the operator A, corresponding to the eigenvalues λ_k with $\operatorname{Re}\lambda_k \leqslant 0$; let P^- be the projector of S_p onto E^-, defined as a segment of the Fourier series in the eigen- and associated vectors. Then in some neighborhood of 0 in S_p $(p \geqslant 3)$ there exists a hypersurface Γ, invariant with respect to N^t, tangent to $E^+ = (I - P^-)S_p$, and filled with trajectories tending to 0 in S_p $(p > 1)$ (and, in the case of sufficiently smooth data, also in $C^{(k)}$) as $t \to \infty$. Γ can be defined by the equation*

$$P^-\mathbf{a} = G(P^+\mathbf{a}), \qquad P^+ = I - P^-,$$

where G is an operator continuous in S_p and $\|G(P^+\mathbf{a})\|_{S_p} = o(\|\mathbf{a}\|_{S_p})$.

If $\operatorname{Re}\lambda_k \neq 0$ $(k = 1, 2, \ldots)$, then every trajectory $N^t\mathbf{a}$ with $\mathbf{a} \notin \Gamma$ leaves some fixed S_p-neighborhood $(p \geqslant 3)$ of zero.

Theorems 1–4 are proved by methods known in the theory of ordinary differential equations and originating in the works of A. M. Lyapunov, O. Perron, and I. G. Petrovskii (see [9,10]), with the use of Lemmas 4 and 5.

6. Together with the results of [11,12], the preceding gives a rigorous proof of the instability of A. N. Kolmogorov’ s spatially periodic flow and of Couette flow between two cylinders for large Reynolds numbers.

7. The results presented here can be obtained for a broad class of parabolic problems. By the same methods one can also consider stability in the spaces $W_p^{(l)}$ and $C^{k,\lambda}$, and also consider continuously acting perturbations.

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Note: Figure translations are in progress. See original paper for figures.