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Abstract

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MATHEMATICS

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**ON THE UNIQUENESS OF THE SOLUTION
OF A TWO-LAYER ONE-PHASE PROBLEM
OF STEFAN TYPE**

(Presented by Academician A. A. Dorodnitsyn, 21 IX 1964)

Below we prove the uniqueness of the solution of the problem*

$$\begin{aligned}
 a^2(x) \frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t}; \quad 0 < x < y(t); \quad x \neq 1; \quad t > 0; \quad u|_{x=y(t)-0} = 0; \\
 \frac{\partial u}{\partial x} \Big|_{x=0} &= 0; \quad u|_{x=1-0} = u|_{x=1+0}; \quad k_1 \frac{\partial u}{\partial x} \Big|_{x=1-0} = k_2 \frac{\partial u}{\partial x} \Big|_{x=1+0}; \quad (1) \\
 u|_{t=0} &= 1; \quad 0 < x < 1;
 \end{aligned}$$

$$\frac{dy}{dt} = -\alpha + \beta \frac{\partial u}{\partial x} \Big|_{x=y(t)}; \quad t > 0; \quad y(0) = 1; \quad (2)$$

$$a(x) = \begin{cases} a_1 = \text{const} > 0; & 0 < x < 1; \\ a_2 = \text{const} > 0; & 1 < x < y; \end{cases} \quad \alpha = \text{const} > 0; \quad \beta = \text{const} < 0 \quad k_i = \text{const} > 0; \quad (i = 1, 2), \quad (3)$$

such that $y(x, t)$ is uniformly bounded and

$$v(t) \equiv \frac{\partial}{\partial x} u \Big|_{x=y(t)} = \frac{w(t)}{\sqrt{t}} = \frac{w(0)}{\sqrt{t}} + v_1(t), \quad (4)$$

where $v_1(t)$ is continuous for $t \geq 0$.

The formulated uniqueness theorem is a consequence of Nirenberg's maximum principle ⁽¹⁾, with Friedman's supplement ⁽²⁾, and of the following lemmas.

Lemma 1. The quantities $w(0)$ and $v_1(0)$ in (4) are determined uniquely, and

$$\beta v_1(0) - \alpha < 0. \quad (5)$$

Lemma 2. Let $u(x, t)$ be a solution of problem (1) for a given $y(t)$ of the form

$$y(t) = 1 + 2\gamma a_2 \sqrt{t} + O(t); \quad \gamma > 0. \quad (6)$$

Then $u^* = \frac{\partial}{\partial x} u$ satisfies the conditions

$$\begin{aligned} a^2(x) \frac{\partial^2 u^*}{\partial x^2} &= \frac{\partial u^*}{\partial t}; \quad 0 < x < y(t); \quad x \neq 1; \quad t > 0; \\ u^*|_{x=0} &= 0; \quad u^*|_{t=0} = 0; \quad 0 \leq x < 1; \quad u^*|_{x=y(t)-0} \leq 0; \\ k_1 u^*|_{x=1-0} &= k_2 u^*|_{x=1+0}; \quad a_1^2 \frac{\partial u^*}{\partial x} \Big|_{x=1-0} = a_2^2 \frac{\partial u^*}{\partial x} \Big|_{x=1+0}, \end{aligned} \quad (7)$$

where

$$j = \lim_{t \rightarrow 0} \sqrt{t} u^*(x(t), t) < 0. \quad (8)$$

* Equations (1), (2) describe the process of crystallization of a melt into which a plate of a more refractory metal is immersed, under the assumption that the melt core is maintained at constant temperature in a state of intensive mixing, that in the boundary layer separating the melt core from the solid phase the temperature is linear, and that the heat flux from the melt core to the plate is prescribed. A two-phase Stefan problem similar in character was recently considered by Tadjbakhsh and Langer⁽³⁾, but they did not address the question of uniqueness of the solution.

along any path of the form

$$x(t) = 1 + c(y(t) - 1); \quad 0 \leq c \leq 1. \quad (9)$$

The limit j from (8) is a continuous function of the parameter c and is attained uniformly. Finally, there exist $\delta > 0$ and $\tau_0 > 0$ such that

$$u^* < 0 \quad \text{for } 1 - \delta < x < 1; \quad 0 < t < \tau_0. \quad (10)$$

We shall prove the uniqueness theorem under the assumption that an admissible solution exists and that Lemmas 1 and 2 hold. Take $\varepsilon > 0$ arbitrarily small and put

$$\begin{aligned} \xi &= 1 + (1 - \varepsilon)(x - 1), \quad \tau = (1 - \varepsilon)^2 t; \\ U(\xi, \tau) &= u(x, t); \quad z(\tau) = 1 + (1 - \varepsilon)(y(t) - 1). \end{aligned} \quad (11)$$

Then $U(\xi, \tau)$ is a solution of problem (1), if in it x, t and y are replaced by ξ, τ and z , respectively, and $x = 0$ by $\xi = \varepsilon$. Condition (2) then becomes

$$\frac{dz}{d\tau} = -\frac{\alpha}{1-\varepsilon} + \beta \frac{\partial}{\partial \xi} U \Big|_{\xi=z(\tau)-0}; \quad z(0) = 1. \quad (12)$$

Let $V(\xi, \tau)$ be the solution of problem (1), if in it ξ, τ and z are written instead of x, t and y , respectively. $V(\xi, \tau)$ satisfies the conditions of Lemma 2. From Lemma 2 it follows, obviously, that there exists $\delta > 0$ such that for every η , $0 < \eta < \delta$, on the arc

$$\Gamma_\eta = \{(\xi - 1)^2 + \tau^2 = \eta^2\}$$

the inequality $u^* < 0$ holds. Let

$$D = \{0 < \xi < z(\tau); 0 < \tau < T; z(\tau) > 1\}; \\ c_\eta = \{(\xi - 1)^2 + \tau^2 < \eta^2\}; \quad D_\eta = D \setminus \bar{c}_\eta.$$

In view of the conditions of Lemma 2 and the remark made, $u^* \leq 0$ on the boundary Γ_η of the domain D_η , while $u^* < 0$ on Γ_η . It follows at once from Friedman's theorem that u^* cannot attain an extremum at $\xi = 1+0$ or $\xi = 1-0$. Consequently, everywhere in D_η , $u^* \leq 0$. But since for $(\xi, \tau) \in \Gamma_\eta$, $u^* < 0$, the strong maximum principle implies that $u^* < 0$ in D_η . Since $\eta > 0$ is arbitrarily small and the arc Γ_η contracts, as $\eta \rightarrow 0$, to the point $\xi = 1, \tau = 0$, it follows that $u^* < 0$ in D . In particular, $u^* < 0$ for $\xi = \varepsilon; \tau > 0$.

Comparing now $U(\xi, \tau)$ and $V(\xi, \tau)$ in the domain

$$D^\varepsilon := \{\varepsilon < \xi < z(\tau); \tau > 0\},$$

we verify, by means of the strong maximum principle and Friedman's theorem, that

$$U(\xi, \tau) \leq V(\xi, \tau) \quad \text{in } D^\varepsilon. \quad (13)$$

Since, moreover, $U|_{\xi=z(\tau)} = V|_{\xi=z(\tau)} = 0$, we have

$$\frac{\partial V}{\partial \xi} \Big|_{\xi=z(\tau)-0} \leq \frac{\partial U}{\partial \xi} \Big|_{\xi=z(\tau)-0}. \quad (14)$$

Finally, from (14) and (2), in view of $\beta < 0$, it follows that

$$\frac{dz}{d\tau} \leq -\frac{\alpha}{1-\varepsilon} + \beta \frac{\partial}{\partial \xi} V \Big|_{\xi=z(\tau)}; \quad \tau > 0. \quad (15)$$

Let now $\tilde{u}(x, t), \tilde{y}(t)$ be a second solution of problem (1), (2). We shall write (ξ, τ) instead of (x, t) . By Lemma 1,

$$\frac{\partial \tilde{u}}{\partial \xi} \Big|_{\xi=\tilde{y}(\tau)} = \frac{w(0)}{\sqrt{\tau}} + \tilde{v}_1(\tau); \quad \tilde{v}_1(0) = v_1(0). \quad (16)$$

Hence, and from (2), we find that

$$\lim_{\tau \rightarrow 0} \left(\frac{dz}{d\tau} - \frac{d\tilde{y}}{d\tau} \right) = \lim_{\tau \rightarrow 0} \left\{ \left[-\frac{\alpha}{1-\varepsilon} + \beta \frac{w(0)}{\sqrt{\tau}} + \frac{\beta v_1(\tau)}{1-\varepsilon} \right] - \left[-\alpha + \beta \frac{w(0)}{\sqrt{\tau}} + \beta \tilde{v}_1(\tau) \right] \right\} = \frac{\varepsilon}{1-\varepsilon} (\beta v_1(0) - \alpha) < 0. \quad (17)$$

Moreover, $z(0) = \tilde{y}(0) = 1$. Thus, there exists $\tau_0 > 0$ such that

$$z(\tau) < \tilde{y}(\tau) \quad \text{for } 0 < \tau < \tau_0. \quad (18)$$

Let $T > 0$ be determined by the condition

$$z(\tau) > 1; \quad \tilde{y}(\tau) > 1 \quad \text{for } 0 < \tau < T. \quad (19)$$

We shall show that (18) holds for all $\tau < T$. Indeed, in the contrary case there must exist $\tau_1 \geq \tau_0$ such that

$$z(\tau) < \tilde{y}(\tau); \quad 0 < \tau < \tau_1; \quad z(\tau_1) = \tilde{y}(\tau_1), \quad (20)$$

and, moreover, for every $\delta > 0$ on the interval $(\tau_1, \tau_1 + \delta)$ there is a τ such that

$$z(\tau) > \tilde{y}(\tau). \quad (21)$$

But this is impossible, since from the maximum principle it follows that

$$\tilde{u}(\xi, \tau) > V(\xi, \tau) \quad \text{for } 0 < \xi < z(\tau); \quad 0 < \tau \leq \tau_1, \quad (22)$$

with $\tilde{v} = V = 0$ for $\xi = z(\tau)$ and $\tau = \tau_1$, and consequently,

$$\frac{\partial \tilde{u}}{\partial \xi} \Big|_{\xi=z(\tau)} \ll \frac{\partial V}{\partial \xi} \Big|_{\xi=z(\tau)} \quad \text{for } \tau = \tau_1, \quad (23)$$

which entails, by virtue of (15) and (2), the inequality

$$\frac{dz}{d\tau} \Big|_{\tau=\tau_1} < \frac{d\tilde{y}}{d\tau} \Big|_{\tau=\tau_1}, \quad (24)$$

Thus, (18) holds for all $\tau \in (0, T)$. Returning to the variables (x, t) , we find with the aid of (11) that

$$\varepsilon + y \left(\frac{t}{(1-\varepsilon)^2} \right) \leq \tilde{y}(t), \quad (25)$$

if

$$\varepsilon + y \left(\frac{t}{(1-\varepsilon)^2} \right) > 1; \quad \tilde{y}(t) > 1. \quad (26)$$

In a completely analogous way we prove the inequality

$$-\varepsilon + y \left(\frac{t}{(1+\varepsilon)^2} \right) \geq \tilde{y}(t). \quad (27)$$

Since $\varepsilon > 0$ is arbitrarily small and T may be regarded as independent of ε , we see that

$$\tilde{y}(t) = y(t), \quad 0 < t < T. \quad (28)$$

But this, by virtue of the uniqueness of the solution of problem (1) for the prescribed $y(t)$, means uniqueness of the solution of the original problem.

In order to verify the validity of Lemmas 1 and 2, let us note that if $G(x, \xi, t)$ is the Green's function of problem (1) on the half-line $x > 0$, i.e., the solution of the problem

$$\begin{aligned} a^2(\xi) \frac{\partial^2 G}{\partial \xi^2} &= \frac{\partial G}{\partial t}; \quad 0 < \xi < \infty; \quad \xi \neq 1; \quad t > 0; \quad \frac{\partial G}{\partial \xi} \Big|_{\xi=0} = 0; \\ k_2 a_1^2 G|_{\xi=1-0} &= k_1 a_2^2 G|_{\xi=1+0}; \quad a_1^2 \frac{\partial G}{\partial \xi} \Big|_{\xi=1-0} = a_2^2 \frac{\partial G}{\partial \xi} \Big|_{\xi=1+0}; \end{aligned} \quad (29)$$

$$G|_{t=0} = \delta(x - \xi),$$

then the solution of problem (1), (2) is representable in the form

$$u(x, t) = \int_0^1 G(x, \xi, t) d\xi + a_2^2 \int_0^t v(\tau) G(x, y(\tau), t - \tau) d\tau. \quad (30)$$

Let g be defined by the conditions

$$\frac{\partial g}{\partial \xi} = -\frac{\partial G}{\partial x}; \quad g|_{\xi=0} = 0. \quad (31)$$

Differentiating (30) for $x \neq 0$, $x \neq 1$, and $x \neq y(t)$, we obtain

$$\frac{\partial u}{\partial x} = -g(x, 1-0, t) + a_2^2 \int_0^t v(\tau) \frac{\partial}{\partial x} G(x, y(\tau), t-\tau) d\tau \quad (32)$$

and in the limit as $x \rightarrow y(t) - 0$

$$w(t) = -2\sqrt{t} g(y(t), 1-0, t) + 2a_2^2 \int_0^t w(\tau) \sqrt{\frac{t}{\tau}} \frac{\partial}{\partial x} G(y(t), y(\tau), t-\tau) d\tau. \quad (33)$$

The functions G and g are easily constructed operationally. It turns out that

$$g(x, 1-0, t) = \begin{cases} (1+\delta)E(x-1, a_2^2 t) + O(e^{-\varkappa/t}), & \text{for } x > 1, \\ (1-\delta)E(x-1, a_2^2 t) + O(e^{-\varkappa/t}), & \text{for } x < 1; \end{cases}$$

$$\frac{\partial}{\partial x} G(x, \xi, t) = \begin{cases} \frac{\partial}{\partial x} E(x-\xi, a_2^2 t) - \delta \frac{\partial}{\partial x} E(x+\xi-2, a_2^2 t) + O(e^{-\varkappa/t}), & \text{for } x > 1, \xi > 1; \\ \frac{1-\delta}{a_1 a_2} \frac{\partial}{\partial x} E\left(\frac{\xi-1}{a_2} + \frac{1-x}{a_1}, t\right) + O(e^{-\varkappa/t}), & \text{for } x < 1, \xi > 1. \end{cases} \quad (33')$$

Here

$$E(x, t) = \frac{e^{-x^2/4t}}{2\sqrt{\pi t}}; \quad \delta = \frac{1-\lambda}{1+\lambda}; \quad \lambda = \frac{k_2 a_1}{k_1 a_2}; \quad \varkappa = \text{const} > 0. \quad (34)$$

All the assertions of Lemmas 1 and 2 follow from (32), (33), and (34) by means of simple, though cumbersome, calculations. In particular, from (33) it follows that

$$\gamma = -\frac{\beta\gamma}{\lambda a_2^2 \sqrt{\pi}} e^{-\gamma^2} - \frac{\gamma}{\lambda} \text{Erf } \gamma; \quad \gamma = \frac{\beta w(0)}{a_2}; \quad (35)$$

$$\beta v_1(0) - \alpha = -\frac{\alpha[1 + I_0(\gamma)]}{1 + I_0(\gamma) - \beta \sum_{i=1}^3 I_i(\gamma)}, \quad (36)$$

where

$$I_0(\gamma) = \frac{\gamma}{\sqrt{\pi}} \int_0^1 \frac{1 - \sqrt{z}}{(1 - z)^{3/2}} e^{-\gamma^2(1 - \sqrt{z})/(1 + \sqrt{z})} dz - \frac{\delta\gamma}{\sqrt{\pi}} \int_0^1 \frac{1 + \sqrt{z}}{(1 - z)^{3/2}} e^{-\gamma^2(1 + \sqrt{z})/(1 - \sqrt{z})} dz,$$

$$-\beta I_1(\gamma) = \gamma^2(1 + \operatorname{Erf} \gamma - \operatorname{Erf} c\gamma),$$

$$-\beta I_2(\gamma) = -\frac{\gamma\delta}{\sqrt{\pi}} \int_0^1 \frac{1 + \sqrt{z}}{\sqrt{z}(1 - z)^{3/2}} e^{-\gamma^2(1 + \sqrt{z})/(1 - \sqrt{z})} \left(\frac{1}{2} - \gamma^2 \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \frac{1 + z}{1 + \sqrt{z}} \right) dz; \quad (37)$$

$$-\beta I_3(\gamma) = \frac{\gamma}{\sqrt{\pi}} \int_0^1 \frac{1}{\sqrt{z}\sqrt{1 - z}} e^{-\gamma^2(1 - \sqrt{z})/1 + \sqrt{z}} \left(\frac{1}{2} - \gamma^2 \frac{1 - \sqrt{z}}{1 + \sqrt{z}} \right) dz.$$

Equalities (35) and (36) may, obviously, be used to construct an approximate solution of problem (1), (2) for small moments of time.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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