SEMIGROUPS OF RECTANGULAR BINARY RELATIONS

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Abstract

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SEMIGROUPS OF RECTANGULAR BINARY RELATIONS

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A binary relation between elements of the sets A and B is any subset of the set $A \times B$. If $A = B$, one speaks of a binary relation on the set A or on the set A. If $\rho \subset A \times B$ and $\sigma \subset B \times C$, then the product of ρ and σ is the binary relation $\sigma \circ \rho \subset A \times C$, where

$$(a, c) \in \sigma \circ \rho \leftrightarrow (\vee b)[(a, b) \in \rho \wedge (b, c) \in \sigma].$$

$\vee$ is the existential quantifier, and $\wedge$ is the symbol of conjunction. If $\rho \subset A \times B$, then the inverse binary relation $\rho^{-1} \subset B \times A$, with $(b, a) \in \rho^{-1} \leftrightarrow (a, b) \in \rho$. If A and B are sets, then $A \mid B$ means that their intersection is empty. The projection of a binary relation ρ onto the set A (onto the set B) will be a subset of the set A (of the set B) and is denoted by $\operatorname{pr}_1 \rho$ ($\operatorname{pr}_2 \rho$). The relation ρ is called **rectangular** if $\rho = \operatorname{pr}_1 \rho \times \operatorname{pr}_2 \rho$. If $\operatorname{pr}_1 \rho = A$, $\operatorname{pr}_2 \rho = B$, then ρ is called a **dense** binary relation.

Let $\rho_i \subset A_i \times B_i$ be a binary relation $(i = 1, 2)$. A pair of mappings (θ_1, θ_2) is called an **isomorphism** of ρ_1 onto ρ_2 if θ_1 is a one-to-one mapping of A_1 onto A_2, θ_2 is a one-to-one mapping of B_1 onto B_2, and for any $a \in A_1$, $b \in B_1$,

$$(a, b) \in \rho_1 \leftrightarrow (\theta_1(a), \theta_2(b)) \in \rho_2.$$

Let us note that if $A_1 = B_1$ and $A_2 = B_2$, the mappings θ_1 and θ_2 need not coincide.

Let G be a partial binary operative (i.e. a nonempty set with a partial binary operation). If $0 \notin G$, then on the set $G^0 = G \cup \{0\}$ one can complete the definition of the operation of the operative G by putting $g_1 g_2 = 0$ if $g_1 g_2$ is not defined in G. If the result is a semigroup, the operative G is called **associative**.

Let ρ be a dense binary relation between elements of nonempty sets A and B. Define on the set $A \times B$ a partial binary operation by putting $(a_1, b_1)(a_2, b_2) = (a_1, b_2)$, the operation being defined if and only if $(a_2, b_1) \in \rho$. The resulting partial binary operative is uniquely determined if ρ is given. We shall denote it by $[\rho]$. It is easy to verify that $[\rho]$ will be associative. The operation in $[\rho]$

will be everywhere defined only if $\rho = A \times B$. In this case, as is not difficult to establish, $[A \times B]$ will be a matrix (or, what is the same, rectangular) band (^{1}), and every matrix band is isomorphic to $[A \times B]$ for suitable A and B.

Associatives of the form $[\rho]$ will be called **matrix**. For example, if $A = B$ and $\rho = \Delta_A$, then $[\Delta_A]$ will be Brandt' s simple groupoid.

A **matrix** (or rectangular) **0-band** (0-bande rectangulaire (^{2})) is any semigroup G with zero having the properties:

1. For any g and g_1 from G, the product gg_1g is equal to 0 or g.
2. For any g_1 and g_2 from G, different from 0, there exists $g \in G$ such that $g_1gg_2 \neq 0$.

Theorem 1. *A semigroup is a matrix 0-band if and only if it is isomorphic to a semigroup of the form $[\rho]^0$, i.e. is obtained from a matrix associative by adjoining a zero.*

Proof. The fact that $[\rho]^0$ will be a matrix 0-relation is verified directly.

Let now G be a matrix 0-relation. Denote the set $G \setminus \{0\}$ by G^- and introduce on this set the binary relations π_1 and π_2:

$$(g_1, g_2) \in \pi_1 \leftrightarrow (\forall g)\,[g_1g = g_2], \qquad (g_1, g_2) \in \pi_2 \leftrightarrow (\forall g)\,[gg_1 = g_2].$$

We shall show that π_1 and π_2 are equivalence relations. If the π_1-class containing the element g is denoted by $\bar{g}$, and the corresponding π_2-class by $\bar{\bar{g}}$, then it is not difficult to show that the mapping $g \to (\bar{g}, \bar{\bar{g}})$ is an isomorphism between G and $[\rho]^0$, where ρ is the binary relation between the elements of the factor sets G^-/π_1 and G^-/π_2, with

$$(\bar{g}_1, \bar{\bar{g}}_2) \in \rho \leftrightarrow g_2g_1 \neq 0.$$

Theorem 1 generalizes the known result on the representation of matrix relations as associatives of the form $[A \times B]$.

Theorem 2. *Let ρ_1 and ρ_2 be dense binary relations, $\rho_1 \subset A_1 \times B_1$ and $\rho_2 \subset A_2 \times B_2$. The following conditions are equivalent:*

1. *ρ_1 and ρ_2 are isomorphic.*
2. *The matrix associatives $[\rho_1]$ and $[\rho_2]$ are isomorphic.*
3. *The matrix 0-relations $[\rho_1]^0$ and $[\rho_2]^0$ are isomorphic.*

This theorem, whose proof we omit, shows that every dense binary relation can be characterized "up to isomorphism" by means of a certain semigroup.

If H is a subset of an associative G and for any $(g_1, g_2) \in H \times H$ for which the product is defined, $g_1g_2 \in H$, then H, with the operation induced on it, is called a **subassociative** of the associative G. Subassociatives of matrix associatives will be called **matrix** (or rectangular) **subassociatives**. A rectangular subassociative will be an associative, but not necessarily a matrix associative. Subsemigroups of matrix 0-relations will be called **matrix** (or rectangular) **0-subrelations**. It is clear that an associative G is isomorphic to a matrix subassociative if and only if G^0 is a matrix 0-subrelation. If G is a partial binary operation, then a binary relation $\nu \subset G \times G$ such that $(g_1, g_2) \in \nu$ if and only if $g_1 g_2$ is defined is called the **domain relation**. It can be shown that *a matrix subassociative G will be a matrix associative if and only if $\nu \circ \nu = G \times G$. A matrix 0-subrelation with zero will be a matrix 0-relation if and only if it is a completely simple semigroup with zero.*

Let ρ be a dense binary relation, $\rho \subset A \times B$, and let σ be a subassociative of the matrix associative $[\rho]$. Then $\sigma \subset A \times B$, i.e. σ is a binary relation. The closedness of σ with respect to the operation from $[\rho]$ means that if $(a_1, b_1) \in \sigma$, $(a_2, b_2) \in \sigma$, and $(a_2, b_1) \in \rho$, then $(a_1, b_2) \in \sigma$. This means that

$$\sigma \circ \rho^{-1} \circ \sigma \subset \sigma. \tag{1}$$

Therefore every matrix subassociative (and hence every matrix 0-subrelation) is uniquely determined by two binary relations ρ and σ satisfying condition (1).

An example of a matrix subassociative is the partial group of Croisot (^{4}), obtained if $A = B$, $\rho = \Delta_A$, and σ is an equivalence relation on A.

A **semigroup of rectangular binary relations** is a semigroup whose elements are rectangular binary relations on some set A, and whose operation coincides with the operation of multiplication of binary relations.

Theorem 3. *In order that a semigroup be isomorphic to a semigroup of rectangular binary relations, it is necessary and sufficient that it be a matrix 0-subrelation.*

Proof. Every semigroup of rectangular binary relations on a set A is a subsemigroup of the semigroup $\Pi(A)$ of all rectangular binary relations. It is easily verified that $\Pi(A)$ is a matrix 0-band.

The second part of the proof follows from the lemma:

Lemma. Let ρ be a binary relation between elements of sets A and B. There exists a set C and mappings θ_1 and θ_2 of, respectively, the sets A and B into the set $\mathfrak{P}(C)$ of subsets of the set C, such that

$$(a, b) \in \rho \leftrightarrow \theta_1(a) \mid \theta_2(b).$$

It is enough to take $C = A \times B$ and define

$$\theta_1(a) = \{a\} \times \rho\langle a \rangle$$

and

$$\theta_2(b) = \rho^{-1}\langle b \rangle \times \{b\}.$$

We note that, by definition,

$$b \in \rho\langle a\rangle \leftrightarrow a \in \rho^{-1}\langle b\rangle \leftrightarrow (a, b) \in \rho.$$

Now let $[\rho]^0$ be a matrix 0-band. Then the mapping $0 \to \emptyset$, $(a, b) \to \theta_1(a) \times \theta_2(b)$ will be, as is easily checked, an isomorphism of the semigroup $[\rho]^0$ onto a semigroup of rectangular binary relations.

By a **transformation** on a set A we shall mean any single-valued binary relation on A (in this case one sometimes speaks of partial transformations). A rectangular transformation is, obviously, the empty binary relation or else a relation of the form $a \times \{a\}$, where $a \subset A$ and $a \in A$. Therefore we shall call rectangular transformations **constant**. It is clear that a semigroup of constant transformations is a special case of a semigroup of rectangular binary relations.

Let $\rho \subset A \times B$ be a binary relation. We shall assume that $A \cap B = \emptyset$. Define $\theta(a) = \rho\langle a\rangle \cup \{a\}$ and to each pair $(a, b) \in A \times B$ assign the constant transformation $\theta(a) \times \{b\}$ on the set $A \cup B$. To zero we assign $\emptyset$. The correspondence obtained will be an isomorphism of the semigroup $[\rho]^0$ onto a certain semigroup of constant transformations. We have proved

Theorem 4. *Matrix 0-bands, and only they, are isomorphic to semigroups of constant transformations.*

We adopt the following notation: let H be a subsemigroup of a semigroup G. Then $H^- = H \setminus \{0\}$, where 0 is the zero of G. If $g \in G$, then $(g)_r$ ($(g)_l$) denotes the principal right (left) ideal generated by the element g, i.e.

$$(g)_r = \{g\}G \cup \{g\}, \qquad (g)_l = G\{g\} \cup \{g\}.$$

Introduce on the set G an equivalence relation π_1, setting $0 \equiv g\ (\pi_1) \leftrightarrow 0 = g$, and for $g_1, g_2 \in G^-$,

$$g_1 \equiv g_2\ (\pi_1)$$

then and only then when there is a chain $(h_1, h_2, \dots, h_n)$ of elements of the semigroup G such that $g_1 = h_1$, $g_2 = h_n$, and

$$(h_1)_r \cdot |(h_2)_r| \cdots |(h_n)_r.$$

In a dual manner (considering left ideals instead of right ones) define π_2. Denote the intersection of the relations π_1 and π_2 by π. We shall say that G is a semigroup **with pseudo-reduction** if $\pi = \Delta_G$, i.e. if from

$$(g_1)_r \cdot | \cdots |(g_2)_r, \qquad (g_1)_l \cdot | \cdots |(g_2)_l$$

it follows that $g_1 = g_2$, whatever $g_1, \dots, g_2$ may be.

The zero 0 is called **almost external** if from $g_1 g_2 g_3 = 0$ it follows that $g_1 g_2 = 0$ or $g_2 g_3 = 0$.

Theorem 5. *For a semigroup G the following properties are equivalent:*

1. *G is a matrix 0-band.*
2. *G is isomorphic to a semigroup of rectangular binary relations.*
3. *G is isomorphic to a semigroup of constant transformations.*
4. *G is a semigroup with pseudo-reduction, containing an almost external zero or containing no zero.*

Proof. It was shown earlier that properties 1–3 are equivalent. Let $[\rho]^0$ be a matrix 0-band. It is easily verified that from

$$(a_1, b_1) \equiv (a_2, b_2)\ (\pi_1)$$

it follows that $a_1 = a_2$, and from

$$(a_1, b_1) \equiv (a_2, b_2)\ (\pi_2)$$

it follows that $b_1 = b_2$. Therefore

$$\pi = \Delta_{(A\times B)^0}.$$

The zero in $[\rho]^0$ will obviously be almost external. It is clear that subsemigroups of a semigroup possessing property 4 also possess these properties. Therefore all matrix 0-bands possess property 4.

Let now G be a semigroup with pseudo-cancellation with an almost external zero. To each $g \in G$ we assign the binary relation $P(g)$ on the set $G \times G$:

$$P(g) = \left(\pi_1\langle g\rangle \times \nu\langle g\rangle^{-1}\right) \times \left(\nu\langle g\rangle \times \pi_2\langle g\rangle\right).$$

It is clear that $P(g)$ is a rectangular relation and that $P(0) = \emptyset$, since $\nu\langle 0\rangle = \emptyset$. From the property of pseudo-cancellation there follows the one-to-one character of the mapping P. Let $g_1g_2 \neq 0$. It is not hard to show that

$$g_1g_2 \equiv g_1(\pi_1), \qquad g_1g_2 \equiv g_2(\pi_2),$$

$$\nu\langle g_1\rangle^{-1} = \nu\langle g_1g_2\rangle^{-1}, \qquad \nu\langle g_2\rangle = \nu\langle g_1g_2\rangle,$$

whence it follows that $P(g_2) \circ P(g_1) = P(g_1g_2)$. If, however, $g_1g_2 = 0$, then $P(g_2) \circ P(g_1) = \emptyset$, and therefore P is an isomorphism of the semigroup G onto a semigroup of rectangular binary relations. Here

$$(g_1, g_2) \in \nu \leftrightarrow g_1g_2 = 0 \vee (g_1 = 0 \wedge g_2 = 0) \vee (g_1 \neq 0 \wedge g_2 = 0).$$

We note that condition 4 is easily written in the form of an infinite system of elementary axioms. It can be shown that *the class of semigroups isomorphic to semigroups of rectangular binary relations (i.e., the class of matrix*

0-subrelations) cannot be characterized by any finite system of elementary axioms.

Of interest is the question of describing all (not necessarily isomorphic) representations of the given semigroup by means of rectangular binary relations, i.e., the question of describing homomorphisms of the semigroup into a matrix 0-relation. The question of homomorphisms of semigroups into matrix 0-relations was considered by G. Lallement and M. Petrich.

An **involuted semigroup of rectangular binary relations** is an involuted semigroup of binary relations (3), all elements of which are rectangular relations.

Theorem 6. *In order that an involuted semigroup be isomorphic to an involuted semigroup of rectangular binary relations, it is necessary and sufficient that for any of its elements* g_1 *and* g_2*, from*

$$g_1 g_1^{-1} = g_2 g_2^{-1} \quad \text{and} \quad g_1^{-1} g_1 = g_2^{-1} g_2$$

there follow $g_1 = g_2$*, and from* $g_1 g_2 \neq 0$ *there follow*

$$g_1 g_1^{-1} = g_1 g_2 g_2^{-1} \quad \text{and} \quad g_2^{-1} g_2 = g_2^{-1} g_1 g_2.$$

Proof of this theorem, analogous to the proof of the preceding theorem but simpler, will be omitted.

We note that a semigroup without zero is isomorphic to a semigroup of rectangular binary relations if and only if it is a rectangular band.

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REFERENCES

1. E. S. Lyapin, *Semigroups*, Moscow, 1960.
2. G. Lallement, Sém. Dubreil-Pisot, 17th year, 1963/64, No. 14, 14—01-14—22.
3. B. M. Schein, DAN, 156, No. 6, 1300 (1964).
4. R. Croisot, C. R., 226, 8, 616 (1948).

Note: Figure translations are in progress. See original paper for figures.