A GENERALIZATION OF E. SKLYARENKO’ S THEOREM

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Abstract

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A GENERALIZATION OF E. SKLYARENKO' S THEOREM

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This work is devoted to a generalization of the well-known theorem of Sklyarenko (1) on the embedding of a normal space into a bicompactum of the same weight and the same dimension (dim). The method of proof of E. G. Sklyarenko makes it possible to prove that, for any countable (or finite) collection of "combinatorial" properties* (see Definition 1) of a space X, there exists a bicompact extension bX of the same weight as X, and with the same collection of "combinatorial" properties (Theorem 1).

We have not been able to show that the property of being acyclic in dimensions $\geq n$ (see Definition 2) is combinatorial. Nevertheless (Theorem 2), for any space X possessing a countable (or finite) collection of combinatorial properties, there exists such a bicompact extension bX of the same weight, with the same collection of combinatorial properties and, moreover, acyclic in all dimensions in which the space X is acyclic. Both of these theorems seem to be of interest even for the classical case of spaces with a countable base.

Definition 1. Let $\mathfrak{B}$ be a certain set of finite complexes. We shall say that a normal space X **has property** $\mathfrak{B}$ if into every finite open cover of the space X one can inscribe a (finite) open cover whose nerve belongs to the set $\mathfrak{B}$. Every property of a normal space equivalent to property $\mathfrak{B}$ for some set $\mathfrak{B}$ of finite complexes will be called **combinatorial**.

The properties of a space X of having dimension not exceeding n ($\dim X \leq n$) and of having density $\leq n$ ($\chi(X) \leq n$) are combinatorial. Similarly one may introduce the following numerical characteristics $\chi_k(X)$. We shall say that $\chi_k(X) \leq n$ if into every finite open cover α of the space X one can inscribe a finite open cover β such that $\chi_k(\beta) \leq n$, where $\chi_k(\beta) \leq n$ means that the number of elements of the cover β entering into the k-th star** of any element of the cover β does not exceed n. By definition $\chi(X) = \chi_1(X)$. The property of a space X of having a given weight τ is not combinatorial, for the reason that the Čech extension of the space X has weight greater than τ, while every finite open cover of the space X extends to a similar cover of the extension βX (with the same nerve), and every finite open cover of the extension βX cuts on X a similar cover.

Theorem 1. *Let two countable (or finite) collections of combinatorial properties*

$\{\mathfrak{B}_k\}$ and $\{\mathfrak{B}'_k\}$ be given; then every normal space X of weight τ, possessing each property $\mathfrak{B}_k$ and possessing none of the properties $\mathfrak{B}'_k$, has a bicompact extension bX of the same weight (as X), possessing all the properties $\mathfrak{B}_k$ and possessing none of the properties $\mathfrak{B}'_k$.

The proof of Theorem 1 follows from the following lemmas.

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* The definition of a combinatorial property was suggested to me by Yu. M. Smirnov; however, the idea of notions of this kind is also encountered in a paper of P. S. Aleksandrov (^{2}).

** The k-th star of an element u of a cover α is the union of all such elements v of the cover α that there exists a chain $u_1, \dots, u_n$ $(n \leq k)$, $u_i \in \alpha$, $u_1 = u$, $u_n = v$, $u_i \cap u_{i+1} \neq \emptyset$.

Lemma 1 (Sklyarenko ([1])). Let X be embedded in I^τ (the Tikhonov cube). On I^τ there exists a uniform structure $\widetilde{\Sigma}_1$ such that it induces on X a structure Σ_1 having the following properties: a) the structure Σ_1 has cardinality τ; b) for every cover α from Σ_1 there are only finitely many covers β from Σ_1 such that $\beta < \alpha$.*

Lemma 2. There exist a cofinal part Σ' of the uniform structure Σ_1 and a system Σ of countable sequences of covers of the space X such that between Σ' and Σ there is a one-to-one correspondence satisfying the following conditions: 1) if α' is from Σ' and $\{a_{ij}\}$ from Σ correspond to one another, then a_{11} is star-refined into α'; 2) if $\alpha' < \beta'$ $(\alpha', \beta' \in \Sigma')$ in the sense of the order of the uniform structure Σ_1, then for the corresponding sequences of covers from Σ we have $a_{ij} < *\beta_{ij}$; 3) the nerves of all covers a_{ij} of the sequence $\{a_{ij}\}$ with fixed index i belong to the family $\mathfrak{B}_i$, and the sequence $\{a_{ij}\}$ is completely ordered as follows:

$$a_{11} < *a_{21} < *a_{12} < *a_{31} < *a_{22} < *a_{13} < *a_{41} < * \dots .$$

Proof. Consider the set S of all systems π of pairs of the form $(\alpha', \{a_{ij}\})$, where $\alpha' \in \Sigma_1$, a_{11} is star-refined into α', $\{a_{ij}\}$ satisfies condition 3), and if $(\alpha', \{a_{ij}\}) \in \pi$, $(\beta', \{\beta_{ij}\}) \in \pi$ and $\alpha' < \beta'$, then $a_{ij} < *\beta_{ij}$. It is easy to see that the set S is nonempty. It is partially ordered: $\pi_1 < \pi_2$ if $\pi_1 \subset \pi_2$. Every ordered subset M of the set S is bounded above by the union of the systems belonging to it. Therefore, by the Hausdorff-Zorn lemma, every system of pairs from S is contained in some maximal system of pairs. Let π^* be a maximal system of pairs. Denote by Σ' the part of the uniform structure Σ_1 through which the covers α' of the system π^* range, and by Σ the corresponding system of sequences $\{a_{ij}\}$. The systems Σ' and Σ are the desired ones. Indeed, conditions 1)-3) are satisfied, and it remains only to prove that Σ' is a cofinal part of Σ_1. Suppose the contrary, i.e., that in the structure Σ_1 there is a cover γ' for which no cover from Σ' follows. In the system Σ' there are only finitely many covers $\alpha^1, \alpha^2, \dots, \alpha^n$ preceding the cover γ'. Let γ_{11} be a cover of property $\mathfrak{B}_1$, star-refined into the cover

$$\gamma' \wedge a_{11}^1 \wedge a_{11}^2 \wedge \cdots \wedge a_{11}^n,$$

where a_{11}^k is from the sequences of covers $\{a_{ij}^k\}$ corresponding to the covers $\alpha^{k'}$; similarly, let γ_{21} be a cover of property $\mathfrak{B}_2$, star-refined into the cover

$$\gamma_{11} \wedge a_{21}^1 \wedge \cdots \wedge a_{21}^n,$$

and so on. Then to the system of pairs π^* one can adjoin the pair $(\gamma', \{\gamma_{ij}\})$, which contradicts the maximality of π^*. The lemma is proved.

Lemma 3. The totality of covers entering into the system Σ forms a uniform structure σ, compatible with the topology of X, satisfying the following conditions: 1) σ has cardinality τ; 2) for every property $\mathfrak{B}_i$ there is in σ a cofinal part consisting of covers whose nerves belong to $\mathfrak{B}_i$.

Proof. From the construction of the system Σ it is clear that conditions 1) and 2) are satisfied. Let us verify whether σ forms a uniform structure. Indeed, for every cover $\alpha \in \{a_{ij}\}$ from Σ there is a cover from Σ star-refined into it. Let $\alpha_{i_1j_1} \in \{a_{ij}\}$ and $\beta_{i_2j_2} \in \{\beta_{ij}\}$; in the product ** $\alpha' \wedge \beta'$, where the cover α' corresponds to $\{a_{ij}\}$, and β' to $\{\beta_{ij}\}$, there is $\gamma' \in \Sigma'$, and consequently a cover $\gamma_{i_3j_3}$ from the sequence $\{\gamma_{ij}\}$, corresponding to γ', such that

$$\alpha_{i_1j_1} < *\alpha_{i_3j_3}, \qquad \beta_{i_2j_2} < *\beta_{i_3j_3},$$

will be refined into $a_{i_1j_1} \wedge \beta_{i_2j_2}$. The structure σ is compatible with the topology of the space X, since for any point $x \in X$ and any of its neighborhoods O_x there exists a cover α' from Σ' such that the star $O_{\alpha'}x$ of the point x with respect to α' is contained in Ox, whence, for the cover α_{ij}, $O_{\alpha_{ij}}x \subseteq Ox$. The lemma is proved.

The bicompact extension bX corresponding to the structure σ has weight τ and possesses each property from the set $\{\mathfrak{B}_k\}$; in order that it possess none of the properties $\mathfrak{B}'_k$, it suffices to take a maximal system π^* such that it contains some pair $(\alpha', \{\alpha_{ij}\})$ satisfying the additional condition: into the cover α_{ij} one cannot insert any cover whose nerve belongs to the set $\mathfrak{B}'_j$.

Corollary. *Let X be a normal space of weight τ, $\dim X = n$, $\chi_k(X) = n_k$. Then there exists a bicompact extension bX of the same weight, of the same dimension, and such that $\chi_k(X) = \chi_k(bX)$ for all k.*

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* For covers α and β from Σ_1 we shall say that α follows from β ($\alpha > \beta$) if and only if, for the covers $\tilde{\alpha}$ and $\tilde{\beta}$ from $\widetilde{\Sigma}_1$ inducing them, the cover $\tilde{\alpha}$ is refined into $\tilde{\beta}$, and covers induced by different covers of the bicompactum I^τ will also be regarded as different.

** $\alpha' \wedge \beta'$ is the cover formed by all possible sets of the form $A \cap B$, where $A \in \alpha'$, $B \in \beta'$.

Indeed, the space X has the combinatorial properties $\dim X \leq n$, $\chi_k(X) \leq n_k$ and does not have the properties $\dim X \leq n-1$, $\chi_k(X) \leq n_k - 1$, i.e., the conditions of Theorem 1 are fulfilled.

Definition 2 (see (3), p. 78). The degree of ∇-cyclicity $\eta_G(X)$ of the space X with respect to the given coefficient group G is the greatest integer n such that $H^n_G(X) \neq 0$, where $H^n_G(X)$ is the n-dimensional cohomology group of the space X, defined by finite covers.

Theorem 2. *Let X be a normal space of weight τ. There exists a bicompact extension bX such that the weight of bX is τ and, for any coefficient group G,*

$$\eta_G(bX) = \eta_G(X)^*.$$

As in the proof of Theorem 1, we shall first prove some lemmas.

Lemma 4. *Let $\{G_i\}$ be a countable (or finite) set of groups, each group G_i consisting of at most countably many elements, $\eta_{G_i}(X) = n_i$. Then, for any sequence of finite open covers $\{\alpha_j; \alpha_j < \alpha_{j+1}\}$, one can construct a sequence of finite open covers $\{\beta_j\}$ satisfying the conditions: $\beta_j < *\beta_{j+1}$, $\alpha_j < *\beta_i$, and, for any group G_i,*

$$\lim_{\{\beta^j\}} H^n_{G_i}(\beta_j) = 0^{**}$$

for all $n > n_i$.

Proof. Since each group G_i is countable, for any finite cover α the group $H^n_{G_i}(\alpha)$ consists of at most countably many elements $h^n_{\alpha k}(G_i)$. If $\eta_{G_i}(X) = n_i$, then for $n > n_i$

$$\lim_{\{\alpha\}} H^n_{G_i}(\alpha) = 0,$$

where $\{\alpha\}$ is the set of all finite open covers; i.e., for any two elements $h^n_{\alpha p}(G_i)$, $h^n_{\beta q}(G_i)$ there is an element $h^n_{\gamma r}(G_i)$ into which they pass under the homomorphisms $H^n_{G_i}(\alpha) \to H^n_{G_i}(\gamma)$, $H^n_{G_i}(\beta) \to H^n_{G_i}(\gamma)$.

Take for β_1 any cover star-refined into α_1; for β_2, one such that it is star-refined into $\beta_1 \wedge \alpha_2$, and the elements $h^{n_1+1}_{\beta_{11}}(G_1)$ and $h^{n_1+1}_{\beta_{12}}(G_1)$, under the homomorphism $H^{n_1+1}_{G_1}(\beta_1) \to H^{n_1+1}_{G_1}(\beta_2)$, pass into one element, and the elements $h^{n_2+1}_{\beta_{11}}(G_2)$, $h^{n_2+1}_{\beta_{12}}(G_2)$, under the homomorphism $H^{n_2+1}_{G_2}(\beta_1) \to H^{n_2+1}_{G_2}(\beta_2)$, also pass into one and the same element of the group $H^{n_2+1}_{G_2}(\beta_2)$. Further, by induction, there exists a cover β_j, star-refined into the cover $\beta_{j-1} \wedge \alpha_j$ and such that, for the groups $G_1, \dots, G_j$, the elements

$$h^{n_i+\nu}_{\beta_{\mu 1}},\ h^{n_i+\nu}_{\beta_{\mu 2}}, \dots, h^{n_i+\nu}_{\beta_{\mu j}}$$

pass under the homomorphisms

$$H^{n_i+\nu}_{G_i}(\beta_\mu) \to H^{n_i+\nu}_{G_i}(\beta_j)$$

into one element $h^{n_i+\nu}_{\beta_j r}(G_i)$ for all $\nu < j$ and $\mu < j$. The sequence of covers thus constructed satisfies the required conditions.

In the following two lemmas it is assumed that the coefficient groups G_i belong to the set $\{G_i\}$ discussed in Lemma 4.

Lemma 5. There exists a cofinal part Σ' of the uniform structure Σ_1 and a system Σ of countable sequences of coverings of the space X such that between Σ' and Σ there is a one-to-one correspondence satisfying the following conditions: 1) if α' from Σ' and $\{\alpha_j\}$ from Σ correspond to each other, then $\alpha_1 > \alpha'$; 2) if $\alpha' < \beta'$, $\alpha', \beta' \in \Sigma'$, in the sense of the order of the uniform structure Σ_1, then for the corresponding sequences $\{\alpha_j\}$ and $\{\beta_j\}$ we have $\alpha_j < *\beta_j$; 3) if $\{\alpha_j\} \in \Sigma$, then $\alpha_1 < *\alpha_2 < \cdots$, and for every group G_i

$$\lim_{\{\alpha_j\}} \overrightarrow{H}^n_{G_i}(\alpha_j) = 0$$

for all $n > n_i$.

The proof of this lemma largely repeats the arguments of Lemma 2.

Lemma 6. The collection of coverings entering the system Σ forms a uniform structure σ, compatible with the topology of X, satisfying the following conditions: 1) σ has cardinality τ; 2) if $G \in \{G_i\}$, then

$$\lim_{\alpha \in \sigma} H^n_G(\alpha) = 0$$

for all $n > \eta_G(X)$.

Proof. We prove assertion 2). Indeed, let

$$h^n_{\alpha_p \mu} \in H^n_G(\alpha_p), \qquad h^n_{\beta_q \nu} \in H^n_G(\beta_q).$$

There is a $\gamma_r \in \{\gamma_j\} \in \Sigma$ such that

$$\alpha_p < *\gamma_r, \qquad \beta_q < *\gamma_r,$$

and since

$$\lim_{\{\gamma_j\}} \overrightarrow{H}^n_G(\gamma_j) = 0,$$

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* If $\eta_G(X) = \infty$, then bX will also have infinite degree of ∇-cyclicity with respect to the coefficient group G.

** $H_G(\beta_j)$ is the n-dimensional cohomology group of the nerve of the cover β_j with coefficient group G_i.

there is a γ_l for which the homomorphisms

$$H_G^n(\alpha_p) \to H_G^n(\gamma_r) \to H_G^n(\gamma_l), \qquad H_G^n(\beta_q) \to H_G^n(\gamma_r) \to H_G^n(\gamma_l),$$

send the elements $h^n_{\alpha_p\mu}$ and $h^n_{\beta_q\nu}$ into one and the same element $h^n_{\gamma_l\lambda}$. The proof of the remaining assertions is analogous to Lemma 3.

Proof of Theorem 2. The bicompact extension bX, corresponding to the structure σ, satisfies the conditions

$$\eta_{G_i}(bX) \leq \eta_{G_i}(X) = n_i.$$

In order that $\eta_{G_i}(bX) = \eta_{G_i}(X) = n_i$, it is necessary to choose the system π^* so that it contains a sequence of coverings $\{\alpha_j\}$ for the nerves of which, in the corresponding groups $H_G^n(\alpha_j)$, there would be nonequivalent elements. This can be achieved, since

$$H_G^{n_i}(X) \neq 0.$$

It remains now for us to make use of the results of M. F. Bokshtein (see (^{3}), Chap. 2), according to which there exists a countable set of groups, each of which is countable, and such that the degree of ∇-cyclicity with respect to any group of coefficients is uniquely determined by the degrees of ∇-cyclicity with respect to groups of coefficients from this set.

Remark. The coverings entering the system Σ could have been chosen not from the set of all finite open coverings, but from any of its cofinal parts. Moreover, for a given countable set of combinatorial properties $\{\mathfrak{B}_i\}$, the sequence $\{\beta_j\}$ mentioned in Lemma 4 can be constructed so that for any property $\mathfrak{B}_i$ in $\{\beta_j\}$ there is an infinite subsequence of coverings whose nerves belong to $\mathfrak{B}_i$, i.e., Theorem 2 can be formulated as follows:

Theorem 1′. For every normal space X possessing a countable set of combinatorial properties $\{\mathfrak{B}_i\}$, there exists a bicompact extension bX of the same weight, with the same set of properties and such that, for any group of coefficients G,

$$\eta_G(X) = \eta_G(bX).$$

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