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Abstract

Full Text

Mathematics

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On the Properties of Sums of Certain Sine and Cosine Series

(Presented by Academician V. I. Smirnov, 21 I 1965)

There exists a sine series $S(x) = \sum_{n=1}^{\infty} b_n \sin nx$ with $b_n > b_{n+1} > 0$ and $b_n \rightarrow 0$ such that $S(x) < 0$ in $(\pi-l, \pi)$, $l > 0$. On the other hand, by Fejér's theorem ⁽¹⁾, $S(x) > 0$ on $(0, \pi)$ if $\Delta^2 b_n \geq 0$ ($n = 1, 2, \dots$) and $b_n \rightarrow 0$ ($b_1 \neq 0$). In ⁽⁴⁾ it was established that $S(x) \geq 0$ on $(0, \pi)$ if $b_n \downarrow 0$ and $\beta_n = \Delta^2 b_n + \Delta^2 b_{n+2} + \dots \geq 0$ ($n = 1, 2, \dots$); if at least one of the numbers β_1 and β_2 is positive, then $S(x) > 0$ on $(0, \pi)$, and a positive lower bound for $S(x)$ on $(0, \pi)$ is given. These results are generalized in the following theorem.

Theorem 1. Let $b_n \downarrow 0$ and

$$\beta_n = \Delta^2 b_n + \Delta^2 b_{n+2} + \Delta^2 b_{n+4} + \dots \geq 0$$

($n = 1, 2, \dots$). Denote by $\{\beta_{n_k}\}$ ($k = 1, 2, \dots$) the subsequence formed from all positive elements of the sequence $\{\beta_n\}$, preserving the former values of the indices n . Then the sum

$$S(x) = \sum_{n=1}^{\infty} b_n \sin nx \geq 0$$

and vanishes on $(0, \pi)$ at exactly those points which are common zeros of the system of functions

$$\left\{ \sin \frac{n_k x}{2} \right\} \quad (k = 1, 2, \dots). \quad (1)$$

The numbers β_n cannot all be equal to zero if $b_1 \neq 0$. Clearly, the system (1) has no common zeros on $(0, \pi)$ if at least one of the numbers β_1 or β_2 is positive, since in this case, respectively, $\sin x/2$ or $\sin x$ has no zeros on $(0, \pi)$. The numbers β_1 and β_2 cannot simultaneously be equal to zero if $b_n \rightarrow 0$, $\Delta^2 b_n \geq 0$, and $S(x) \neq 0$.

We note that there exists a sine series whose coefficients satisfy the conditions of Theorem 1, and whose sum vanishes at a point lying inside the interval $(0, \pi)$.

The proof of Theorem 1 is based on the representation

$$S(x) = 2 \operatorname{ctg} \left(\frac{x}{2} \right) \sum_{k=1}^{\infty} \beta_n \sin^2 \left(\frac{n_k x}{2} \right), \quad (2)$$

which is derived from formula (5) (5).

If $\beta_n \geq 0$ for $n \geq N + 1$, then for any $m \geq N$ and $0 < x < \pi$ it is true that

$$S(x) \geq 2 \operatorname{ctg} \left(\frac{x}{2} \right) \sum_{n=1}^m \beta_n \sin^2 \left(\frac{n}{2} x \right). \quad (3)$$

Let, for example, $b_n = 1/n^2$. Then all $\beta_n > 0$. For definiteness, putting $m = 2$ in (3), we obtain:

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2} > \left(\frac{\pi^2}{6} - 1 \right) \sin x + \left(7 - \frac{2}{3}\pi^2 \right) \sin x \cos^2 \left(\frac{x}{2} \right), \quad 0 < x < \pi.$$

Consider the cosine series $C_1(x) = \sum_{n=1}^{\infty} a_n \cos(2n-1)x$ (it may diverge at $x = 0$). The function $C_1(x)$ is even; moreover, $C_1(x) = -C_1(\pi - x)$. Therefore it suffices to study $C_1(x)$ for $0 < x < \pi/2$. If $\Delta^3 a_n \geq 0$ ($n = 1, 2, \dots$) and $a_n \rightarrow 0$, then it is known ⁽¹⁾ that $C_1(x) > 0$ on $(0, \pi/2)$. (Moreover, $C_1(x)$ decreases monotonically on $(0, \pi/2)$.) On the other hand, there exists a cosine series $\sum_{n=1}^{\infty} a_n \cos(2n-1)x$ with $\Delta^2 a_n \geq 0$ and $a_n \rightarrow 0$, whose sum $C_1(x) < 0$ on $(\pi/2 - \varepsilon, \pi)$, $\varepsilon > 0$. In ⁽⁶⁾ it was established that $C_1(x) \geq 0$ on $(0, \pi/2)$, if $a_n \downarrow 0$ and

$$\Delta^3 a_n + \Delta^3 a_{n+2} + \Delta^3 a_{n+4} + \dots \geq 0 \quad (n = 1, 2, \dots).$$

In Theorem 2 these results are generalized.

Theorem 2. Let $a_n \downarrow 0$ and

$$\alpha_n = \Delta^3 a_n + \Delta^3 a_{n+2} + \Delta^3 a_{n+4} + \dots \geq 0$$

($n = 1, 2, \dots$). Denote by $\{a_{n_k}\}$ ($k = 1, 2, \dots$) the subsequence consisting of all positive elements of $\{a_n\}$, with the former values of the indices n preserved. Then the sum $C_1(x) = \sum_{n=1}^{\infty} a_n \cos(2n-$

$$-1)x \geq 0$$

and vanishes on $(0, \pi/2)$ at those and only those points which are common zeros of the system

$$\{\sin n_{kx}\} \quad (k = 1, 2, \dots). \quad (4)$$

In particular, if at least one of the numbers α_1 or α_2 is positive, then the system (4) has no common zeros on $(0, \pi/2)$, since neither $\sin x$ nor $\sin 2x$ has zeros on $(0, \pi/2)$. The numbers α_1 and α_2 cannot vanish simultaneously if $\Delta^3 a_n \geq 0$ ($n = 1, 2, \dots$), $a_n \rightarrow 0$, and $C_1(x) \neq 0$.

Theorem 2 is proved on the basis of the transformation

$$C_1(x) = \cos x \sum_{k=1}^{\infty} \alpha_{n_k} \frac{\sin^2(n_{kx})}{\sin^2 x}. \quad (5)$$

Theorem 3. If $a_n \rightarrow 0$, $\Delta^2 a_n \geq 0$,

$$p_n = \sum_{k=1}^{\infty} (k+1) \Delta^4 a_{n+2k} \geq 0$$

for $n \geq N+1$, then for any integer $m \geq N$

$$C(x) \geq C(\pi) + 2 \operatorname{ctg}^2 \left(\frac{x}{2} \right) \sum_{n=1}^m p_n \sin^2 \left(\frac{n}{2} x \right), \quad (6)$$

where

$$C(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

If $m = \infty$, then (6) becomes an equality.

The proof of Theorem 3 is based on the transformation

$$C(x) = C(\pi) + \operatorname{ctg} \left(\frac{x}{2} \right) \sum_{n=1}^{\infty} \Delta d_n \sin nx,$$

where $\Delta d_n = \Delta^2 a_n + \Delta^2 a_{n+2} + \dots$ ⁽⁵⁾, with the use of (3).

The conditions of Theorem 3 are certainly fulfilled if $a_n \downarrow 0$ and

$$\Delta^4 a_n + \Delta^4 a_{n+2} + \Delta^4 a_{n+4} + \dots \geq 0$$

($n \geq N+1$).

Let $N = 0$. Then $p_n \geq 0$ ($n = 1, 2, \dots$). If $C(x) \neq \text{const}$, then not all p_n can be equal to zero. In this case $C(x)$ attains its least value at the points $x = \pi \pmod{2\pi}$. Consequently, $C(x) \geq C(\pi)$. This result is known ⁽¹⁾, but under stronger restrictions: $\Delta^4 a_n \geq 0$ ($n = 1, 2, \dots$) and $a_n \rightarrow 0$.

It is known ⁽³⁾ that $C(x) \geq 0$, if $\Delta^2 a_n \geq 0$ ($n = 0, 1, 2, \dots$) and $a_n \rightarrow 0$. Under the conditions of Theorem 3 (for $N = 0$) it turns out that $C(x) \geq 0$ also for

$\Delta^2 a_0 < 0$, but nevertheless it must be that $\Delta^2 a_0 > -(\Delta^2 a_2 + \Delta^2 a_4 + \Delta^2 a_6 + \dots)$ (the expression in parentheses is positive, if the case $a_n \equiv 0$ ($n = 2, 3, \dots$) is excluded).

Remark. Let $a_n \rightarrow 0$, $\Delta^2 a_n \geq 0$, and $p_n \geq 0$ for $n = 2, 3, \dots$. Then, for any integer $k \geq 1$,

$$C(x) \leq C(\pi) + \operatorname{ctg} \left(\frac{x}{2} \right) \sum_{n=1}^{\infty} \Delta a_n \sin nx - 2 \operatorname{ctg}^2 \left(\frac{x}{2} \right) \sum_{n=1}^k p_{n+1} \sin^2 \left(\frac{n}{2} x \right).$$

If $k = \infty$, then this inequality becomes an equality.

Theorem 4. If $b_n \rightarrow 0$ and $\Delta^2 b_n \geq 0$ for $n \geq N + 1$, then for any integer $m \geq N$ and $0 < x < \pi$,

$$S_1(x) \geq \frac{b_1}{2} \sin x + \frac{1}{2 \sin x} \sum_{n=1}^m (\Delta^2 b_n + 4\beta_{n+1} \cos^2 x) \sin^2(nx), \quad (7)$$

where

$$S_1(x) = \sum_{n=1}^{\infty} b_n \sin(2n-1)x \quad \text{and} \quad \beta_{n+1} = \Delta^2 b_{n+1} + \Delta^2 b_{n+3} + \Delta^2 b_{n+5} + \dots$$

If $m = \infty$, then (7) becomes an equality.

For example, putting $m = 1$ in (7), we find

$$S_1(x) \geq \frac{b_1}{2} \sin x + \frac{1}{2} (\Delta^2 b_1 + 4\beta_2 \cos^2 x) \sin x \quad (0 \leq x \leq \pi).$$

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REFERENCES

- ¹ L. Fejér, Trans. Am. Math. Soc., **39**, 18 (1936).
- ² N. K. Bari, *Trigonometric Series*, Moscow, 1961.
- ³ W. H. Young, Proc. London Math. Soc., **12**, 41 (1913).
- ⁴ I. N. Pak, DAN, **151**, No. 1, 38 (1963).
- ⁵ I. N. Pak, Vestn. Leningrad Univ., No. 7 (1963).
- ⁶ I. N. Pak, Tr. uchebn. institutov svyazi, issue 12 (1963).

Note: Figure translations are in progress. See original paper for figures.

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