ON THE EIGENFUNCTIONS OF THE EQUATION $\Delta u+\lambda f(u)=0$

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Abstract

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MATHEMATICS

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ON THE EIGENFUNCTIONS OF THE EQUATION $\Delta u + \lambda f(u) = 0$

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We consider the question of the existence and the question of the absence of eigenfunctions of the boundary-value problem

$$\Delta u + \lambda f(u) = 0, \qquad u|_\Gamma = 0, \tag{1}$$

where Δ is the Laplace operator and $f(0) = 0$. The bounded domain G of n-dimensional Euclidean space has boundary $\Gamma \in C_{0,1}$.

In the case $n = 2$, the existence of eigenfunctions of problem (1), under certain conditions on the function $f(u)$, was proved in works ([1,2]). These results are strengthened in Theorem 3. In the case $n \geq 2$, the existence of eigenfunctions of problem (1) was proved for monotone and "concave" functions, or functions close to them, $f(u)$, in ([3]).

Put

$$F(u) = \int_0^u f(t)\,dt.$$

Theorem 1. *Let $n > 2$, and suppose there exists a function $v(x) \in \dot{W}_2^1(G)$ for which*

$$\int_G F(v)\,dx = \mu \neq 0.$$

Let the function $f(u)$ from $C_{0,\alpha}$ satisfy the condition

$$|f(u)| \leq A + B|u|^m, \qquad m < (n+2)/(n-2), \tag{2}$$

where A and B are arbitrary constants.

Then the boundary-value problem (1) has an eigenfunction $\varphi(x) \in C_2(G) \cap C_0(\bar{G})$ and

$$\int_G F(\varphi)\,dx = \mu.$$

We give the scheme of the proof. Consider the variational problem of finding the minimum of the functional

$$E(\psi) = \int_G \sum_{i=1}^{n} (D_i\psi)^2 \, dx$$

in the class of functions $\psi(x) \in \dot{W}_2^1(G)$, under the condition

$$\int_G F(\psi)\, dx = \mu.$$

Under these conditions, by inequality 2, the functional $E(\psi)$, as is known ([4]), has $\inf E(\psi) = l > 0$, and a minimizing sequence of the functional $E(\psi)$ contains a subsequence $\{\psi_k\}$ converging strongly in the space $L_q(G)$, $(q < 2n/(n-2))$, to a function $\varphi(x)$, for which we have

$$\int_G f(\psi_k)\psi_k \, dx \to \int_G f(\varphi)\varphi \, dx \quad \text{as } k \to \infty; \qquad \int_G F(\varphi)\, dx = \mu.$$

The limiting function $\varphi(x)$ belongs to $\dot{W}_2^1(G)$, and $E(\varphi) = l$. To prove these facts it suffices to use the theorem of S. L. Sobolev ([4], p. 42) and the weak compactness in $\dot{W}_2^1(G)$ of the sequence $\{\psi_k\}$, bounded in $\dot{W}_2^1(G)$.

The function $\varphi(x)$ satisfies the integral identity

$$\int_G f(\varphi)\varphi \, dx \cdot \int_G \sum_{i=1}^{n} D_i\varphi \cdot D_i\xi \, dx - E(\varphi) \int_G f(\varphi)\xi \, dx = 0 \qquad (3)$$

with respect to functions $\xi(x) \in \dot{W}_2^1(G)$.

For the function $\varphi(x)$ the inequality

$$\int_G |f(\varphi)| \, dx \neq 0 \qquad (4)$$

holds.

This inequality is proved on the basis of Green' s formula

$$n \int_G F(\eta)\, dx + \int_G f(\eta) \sum_{i=1}^{n} x_{iD} i\eta \, dx = 0,$$

valid for essentially bounded functions $\eta(x) \in \dot{W}_2^1(G)$.

From inequality (4) and relation (3) it follows that $\int_G f(\varphi)\varphi\,dx \neq 0$. Denoting

$$\lambda = E(\varphi) \Big/ \int_G f(\varphi)\varphi\,dx,$$

we rewrite relation (3) in the form

$$\int_G \sum_{i=1}^{n} D_i\varphi \cdot D_i\xi\,dx - \lambda \int_G f(\varphi)\xi\,dx = 0. \tag{5}$$

Choosing as the function $\xi(x)$ in this relation a smoothly truncated Green function $K_\varepsilon(x, y)$ corresponding to zero boundary values, and passing to the limit, we obtain

$$\varphi(x) = \lambda \int_G K(x, y) f[\varphi(y)]\,dy. \tag{6}$$

This equation, which is satisfied by the function $\varphi(x) \in \dot{W}_2^1(G)$, holds almost everywhere.

Using embedding theorems and successively applying the properties of integrals of potential type, taking inequality (2) into account, from equation (6) we obtain the assertion of the theorem.

Remark. As shown below, under the conditions of the theorem the exponent m cannot be replaced by $m \geq (n+2)/(n-2)$.

Theorem 2. *Suppose the conditions of Theorem 1 are satisfied. Then the boundary-value problem (1) has a continuum of eigenfunctions*

$$\varphi'(x) \in C_2(G) \cap C_0(\overline{G}),$$

and

$$\int_G F(\varphi')\,dx = \mu',$$

where μ' is any number sufficiently close to μ.

For the proof it is enough to note that, by virtue of inequality (4), for the function $\varphi(x) \in \dot{W}_2^1(G)$ obtained above, there exists a function $\xi_0(x) \in \dot{W}_2^1(G)$ such that

$$\int_G f(\varphi)\xi_0\,dx \neq 0.$$

Consequently, the equation

$$\int_G F(\varphi + \varepsilon\xi_0)\,dx = \mu'$$

is solvable with respect to ε for any μ' sufficiently close to μ.

Before turning to the case $n = 2$, we formulate a lemma.

Lemma 1. *For $n = 2$ there is a completely continuous embedding of $W_2^1(G)$ into the Orlicz space $L_M(G)$ corresponding to the Young function*

$$M(u) = |u|^b e^{c|u|^a},$$

where $b > 1$, $c > 0$ are arbitrary constants and a is any quantity satisfying the condition $0 \leq a < 2$.

* More precisely, into the subspace $E_M(G)$, which is the closure in the norm of $L_M(G)$ of the set of bounded

Theorem 3. Let $n = 2$, and suppose there exists a function $v(x) \in \overset{\circ}{W}{}_2^1(G)$ such that

$$\int_G F(v)\, dx = \mu \neq 0.$$

Let the function $f(u)$ from $C_{0,\alpha}$ satisfy the condition

$$|f(u)| \leq A + B|u|^b e^{c|u|^a},$$

where $a < 2$; A, B, b, c are arbitrary constants. Then the boundary-value problem (1) has an eigenfunction

$$\varphi(x) \in C_2(G) \cap C_0(\overline{G}),$$

and moreover

$$\int_G F(\varphi)\, dx = \mu.$$

The proof of this theorem is analogous to the proof of Theorem 1, if one uses Lemma 1 and the continuity condition for the Nemytskii operator in the Orlicz space (1).

Remark. The assertion of Theorem 2 is valid under the conditions of Theorem 3.

A sufficient criterion for the absence of eigenfunctions (in the class of real functions) of the boundary-value problem (1) is established on the basis of the following lemma.

Lemma 2. Let $u(x)$ be a solution of the boundary-value problem (1), belonging to the class

$$W_2^2(G) \cap C_0(\overline{G}), \qquad \Gamma \in C_{1,\beta}.$$

Then for the function $u(x)$ the formula

$$\lambda n \int_G F(u)\,dx + \frac{2-n}{2}\lambda \int_G f(u)u\,dx = \frac{1}{2}\int_\Gamma u_\nu^2(\mathbf{r}\cdot\vec{\nu})\,ds. \tag{7}$$

holds.

Here $(\mathbf{r}\cdot\vec{\nu})$ is the scalar product of the unit outward normal $\mathbf{n}$ to the boundary Γ and $\mathbf{r}$, the radius vector of the boundary point with respect to some fixed point of G; u_ν is the derivative in the direction of the normal $\vec{\nu}$.

To prove this formula it suffices to apply the Gauss-Ostrogradsky formula

$$\int_G \operatorname{div}\mathbf{P}\,dx = \int_\Gamma P_\nu\,ds$$

to the function

$$\mathbf{P}(x) = \sum_{i=1}^n x_{iD} iu \operatorname{grad} u$$

and to use Green' s formula

$$\int_G \sum_{i=1}^n (D_i u)^2\,dx = \lambda \int_G uf(u)\,dx. \tag{8}$$

For a domain $G \in C_{1,\beta}$, star-shaped with respect to some point ($(\mathbf{r}\cdot\vec{\nu}) > 0$ almost everywhere on Γ), from Lemma 2 we obtain a sufficient condition for the absence of eigenfunctions of problem (1) when $f(u) \in C_{0,\alpha}$.

Let

$$\lambda\left[\frac{n-2}{2n}uf(u) - F(u)\right] > 0 \quad \text{for } u \neq 0.$$

Then there is no eigenfunction $\varphi(x) \in W_2^2(G) \cap C_0(\overline{G})$ of problem (1).

We note that the sign of λ coincides with the sign of the integral

$$\int_G uf(u)\,dx$$

by virtue of formula (8).

In the case $f(u) \geq 0$ for $u \geq 0$ we obtain a sufficient criterion for the absence of positive eigenfunctions of problem (1) when $f(u) \in C_{0,\alpha}$.

Let

$$\frac{n-2}{2n} u f(u) - F(u) \geq 0 \quad \text{for } u \geq 0.$$

Then there is no positive eigenfunction $\varphi(x) \in W_2^2(G) \cap C_0(G)$ of problem (1).

Example. The boundary-value problem

$$\Delta u + \lambda |u|^m = 0, \qquad u|_\Gamma = 0$$

in a domain $G \in C_1$ star-shaped with respect to a point has no eigenfunction $u(x) \in C_2(G) \cap C_0(\overline{G})$ for $m \geq (n+2)/(n-2)$, $n > 2$.

Suppose the contrary. Put $u(x) = kv(x)$, where $k = |\lambda|^{-1/(m-1)} \operatorname{sign} \lambda$ $(\lambda \neq 0)$. Then the function $v(x)$ satisfies the equation $\Delta v + |v|^m = 0$ and $v|_\Gamma = 0$. By the maximum principle the function $v(x) \geq 0$ and, consequently, satisfies the equation $\Delta v + v^m = 0$. The function $v(x)$, of class $C_2(G) \cap C_0(\overline{G})$, by virtue of the equation belongs to $W_2^2(G)$ and

$$\int_\Gamma v v_\nu^2 \, ds \neq 0,$$

since

$$\int_\Gamma v_\nu \, ds = -\int_G v^m \, dx < 0$$

by the assumption. Applying formula (7) to the function $f(v) = v^m$ and $\lambda = 1$, we obtain a contradiction.

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REFERENCES

[1] M. A. Krasnosel' skii, Ya. B. Rutitskii, *Convex Functions and Orlicz Spaces*, Moscow, 1958.
[2] N. Levinson, Arch. Rat. Mech. and Analysis, **11**, No. 3 (1962).
[3] M. A. Krasnosel' skii, V. Ya. Stetsenko, *Siberian Mathematical Journal*, **4**,

No. 1, 120 (1963).
[4] S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, Novosibirsk, 1962.
[5] E. Gagliardo, *Amer. Math. Soc. Transl. Ser. 2*, **5**, 4, 87 (1961).

Note: Figure translations are in progress. See original paper for figures.