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Abstract

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MATHEMATICS

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ON THE TOPOLOGICAL STRUCTURE OF STABILITY REGIONS OF A SELF-ADJOINT SYSTEM OF DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

(Presented by Academician I. G. Petrovskii on 5 XI 1964)

Consider a real linear system of $2k$ differential equations

$$Q(t)\frac{dy}{dt} - \left\{S(t) - \frac{1}{2}\frac{d}{dt}Q(t)\right\} y = 0, \tag{1}$$

where $Q(t)$ is a nonsingular skew-symmetric matrix, $Q'(t) = -Q(t)$, and $S(t)$ is a symmetric matrix, $S'(t) = S(t)$, with elements depending periodically on the parameter t, $Q(t+\omega) \equiv Q(t)$, $S(t+\omega) \equiv S(t)$. The matrices of functions $S(t)$ and $\dot{Q}(t)$ are assumed to be piecewise continuous.

System (1) is a generalization of a Hamiltonian system of equations*. The latter is obtained when

$$Q(t) = I = \begin{pmatrix} 0 & E_k \\ -E_k & 0 \end{pmatrix} \tag{2}$$

(E_k is the identity matrix of order k).

We shall denote the matriciant of system (1) by $Y(t)$ ($Y(0) = E_{2k}$). System (1) is called **stable** if all its solutions are bounded as $t \to \infty$. System (1) is called **strongly stable** if it is stable and all sufficiently close systems of the same form (cf. (1)) also possess this property.

N. Levinson, who introduced systems (1) into consideration and investigated their stability, showed in his work (^{2}) that the known sufficient conditions for strong stability for a Hamiltonian system (see (^{1,3})), carry over without change also to systems of the form (1). It can be proved that these conditions are also necessary for strong stability.

Thus, system (1) is strongly stable if and only if all root subspaces of the monodromy matrix $Y(\omega)$ are definite in the metric determined by the Hermitian form

$$i(If, f) = i\sum_{s=1}^{k}(f_{s+k}\bar{f}_s - f_s\bar{f}_{s+k}),$$

in other words, when there are no coincident multipliers of different kind.

In the present work the structure of the stability regions of systems (1) is investigated and, in contrast to Hamiltonian systems, it is shown that for $k > 1$ there exists only a finite number of classes of topologically equivalent strongly stable systems**. Namely, the following assertion is valid:

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* It is not difficult to show that the left-hand side of (1) is a general formally self-adjoint differential operator of first order in the real space of vector functions $y = (y_1, y_2, \dots, y_{2k})$.

** For $k = 1$ system (1) is Hamiltonian.

Theorem. The set of all strongly stable systems for $k > 1$ splits into 2^{k+1} connected components. Each component is characterized by one of the 2^k types of arrangement of the multipliers of the first and second kind on the unit circle and, in addition, by one of the two possible values of the index: 0 or 1.

We outline the proof of the theorem. Without loss of generality, one may assume that in (1)

$$Q(0) = Q(\omega) = I \tag{3}$$

(this can always be achieved by a substitution).

Let

$$Y(t) \qquad (0 \le t \le \omega) \tag{4}$$

be a nonsingular continuous piecewise-differentiable function satisfying the condition $Y(0) = E_{2k}$.

Lemma. In order that the matrix-function (4) be a matrizant of a system of the form (1), it is necessary and sufficient that

$$Y^*(\omega)IY(\omega) = I, \tag{5}$$

i.e., that the end of the curve $Y(\omega)$ be a symplectic matrix.

The necessity of condition (5) follows from the work of N. Levinson (^{2}). Let us prove its sufficiency. Put

$$Q(t) = Y^{*-1}(t) I Y^{-1}(t). \tag{6}$$

It is easy to see that $Q(t)$ is a skew-symmetric nonsingular matrix satisfying condition (3) and the required smoothness conditions. We next note that

$$Y^*(t)Q(t)Y(t) \equiv I. \tag{7}$$

Differentiating the identity (7), we obtain

$$\dot{Y}^*QY + Y^*\dot{Q}Y + Y^*Q\dot{Y} = 0. \tag{8}$$

Hence

$$\dot{Q}Y - \{-Y^{*-1}\dot{Y}^*Q - \dot{Q}\}Y = 0. \tag{9}$$

For the proof of the lemma it remains to show that the skew-symmetric part of the matrix in braces is equal to $-\frac{1}{2}\dot{Q}$. But, by virtue of (8),

$$\frac{1}{2}[-Y^{*-1}\dot{Y}^*Q - Q\dot{Y}Y^{-1}] = \frac{1}{2}\dot{Q}.$$

Thus the lemma is proved completely.

The lemma makes it possible to reduce the question of deformation of systems of the form (1) to the deformation of the corresponding matrizants. As was proved in (^{1}) (see also (^{5})), symplectic matrices of stable type with the same arrangement of multipliers form a connected, simply connected component. Therefore the ends of two curves $Y_1(t)$ and $Y_2(t)$ $(0 \leq t \leq \omega)$ with the same arrangement of multipliers can always be joined by a curve lying entirely in the corresponding component, and any two joining curves are homotopic.

It remains only to establish whether the closed curve arising as a result of joining the ends is contractible to a point or not. The representation of the matrix-function in polar form $Y(t) = H(t)U(t)$ makes it possible to reduce the question to consideration of the closed projection $U(t)$ in the group O_{2k}^+ of real orthogonal matrices of order $2k$ with determinant 1. The latter, as is known, is not simply connected (see (^{4}), p. 360) and contains a cycle C_0, not homotopic to zero, with every cycle in O_{2k}^+ either homotopic to zero or homotopic to C_0.

Thus, the class of matrizants with the same arrangement of multipliers is divided into two subclasses. In order to determine to which subclass a given matrizant belongs, it is sufficient join its endpoint $Y(\omega)$ to the identity matrix by a curve lying in the component of symplectic matrices of stable type, and determine whether the cycle thus obtained is homotopic to zero or not. In the first case we assign to the matricant the index 0, and in the second, 1.

In conclusion we give the following result, which makes it possible to find the value of the index.

Theorem 2. Let

$$U(t) = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \qquad (0 \leqslant t \leqslant \omega) \tag{10}$$

be a closed curve in the group O_{2k}^+ $(k > 1)$, and let

$$\Delta(t) = \det[u_1 + u_4 + i(u_2 - u_3)] \neq 0. \tag{11}$$

Put

$$\operatorname{Arg} \Delta(t)\Big|_0^\omega = 2p\pi. \tag{12}$$

Then, in order that the cycle (10) be homotopic to zero, it is necessary and sufficient that the number p in formula (12) be even.

This theorem was communicated to the authors by F. A. Berezin, who obtained it while studying the spinor representation of the group O_{2k}^+ (see [6]). Below we give an elementary proof of the theorem.

Introduce the unitary matrix (cf. [1], p. 25)

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} E_k & E_k \\ iE_k & -iE_k \end{pmatrix} \tag{13}$$

and put

$$\widetilde{U} = M^* U M = \begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix}. \tag{14}$$

Here, as is easy to verify, $\Phi = \frac{1}{2}[u_1 + u_4 + i(u_2 - u_3)]$ (cf. (11)).

We note in passing that any unitary matrix of the form (14) is transformed by M^{-1} into a real orthogonal matrix. By virtue of the unitarity of $\widetilde{U}$ we have: $\overline{\Psi}'\Phi + \Phi'\overline{\Psi} = 0$, whence it follows that the matrix

$$C = \Psi\Phi^{-1}$$

is complex skew-symmetric: $C' = -C$.

This circumstance makes it possible to represent $\widetilde{U}$ in the form of the following product of two unitary matrices:

$$\widetilde{U} = \widetilde{U}_I \widetilde{U}_{\mathrm{II}} = \begin{pmatrix} H & C\overline{H} \\ CH & \overline{H} \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & \overline{V} \end{pmatrix}, \tag{15}$$

where $H = (E + CC^*)^{-1/2}$, and V is a unitary matrix, $\Phi = HV$. Contracting the curve $C(t)$ in the Euclidean space of skew-symmetric matrices to zero, we contract $\widetilde{U}_I(t)$ to E_{2k}. After this we represent $\widetilde{U}_{\mathrm{II}}$ in the form

$$\widetilde{U}_{\mathrm{II}} = \begin{pmatrix} V_1 & 0 \\ 0 & \overline{V}_1 \end{pmatrix} \begin{pmatrix} V_0 & 0 \\ 0 & \overline{V}_0 \end{pmatrix}, \tag{16}$$

where V_1 is a unitary unimodular matrix, and V_0 is diagonal, its upper-left element being equal to $\det V$, and all the others to one. The left factor in (16) is always contracted to a point, since the group of unitary unimodular matrices is simply connected (see [4], p. 360); the right factor—in view of the fact that

$$\operatorname{Arg} V\Big|_0^\omega = \operatorname{Arg} \Phi\Big|_0^\omega = 2p\pi$$

—only for even p (see [4], p. 361). Thus Theorem 2 is proved.

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Note: Figure translations are in progress. See original paper for figures.

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