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Abstract

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GENERAL BOUNDARY-VALUE PROBLEMS FOR PARABOLIC EQUATIONS IN A CLOSED DOMAIN

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We shall consider, in a bounded closed domain G of the space $(t, x_1, \dots, x_n)$, the equation

$$L(t, x, \partial/\partial t, \partial/\partial x)u = f, \tag{1}$$

where L is a certain polynomial in the operators $\partial/\partial t$, $\partial/\partial x$ with coefficients depending on $t, x_1, \dots, x_n$. The principal part of the operator L is defined as follows.

Let l be the maximal order of the derivatives with respect to t entering the left-hand side of equation (1); $2pl$ the maximal order of the derivatives with respect to x. We shall call the generalized order of the derivative $\partial^{k_1+k_2}/\partial t^{k_1}\partial x^{k_2}$ the number $2pk_1 + k_2$. Let the highest generalized order of the derivatives entering L be $2pl$. We shall call the principal part of the operator L the operator $L_0(t, x, \partial/\partial t, \partial/\partial x)$, which is obtained from L by discarding terms whose generalized order is less than $2pl$. The number $2pl$ will be called the generalized order of equation (1). It is assumed that $L_0(t, x, i\tau, i\sigma) \neq 0$ if $|\tau| + |\sigma| \neq 0$ and τ, σ are real.

The boundary Γ of the domain G is assumed to be an infinitely differentiable surface. We suppose that on Γ there exist only two points (t^0, x^0) and (t^1, x^1) at which the tangent plane is perpendicular to the t-axis. Let $t^0 = 0$, $x^0 = 0$. We assume that, in a neighborhood of $(0, 0)$, the equation of Γ has the form

$$t^q = \sum_{i_1+\dots+i_n=2pq} a_{i_1\dots i_n} x_1^{i_1} \cdots x_n^{i_n} + o(|x|^{2pq}), \qquad q \text{ an integer.} \tag{2}$$

On Γ there are prescribed pl boundary conditions of the form

$$B_j(t, x, \partial/\partial t, \partial/\partial x)u = \varphi_j, \tag{3}$$

where B_j are differential operators of generalized order m_j.

The coefficients of the operators L, B_j are infinitely differentiable everywhere except at the special points. At the point $(0, 0)$ we assume that the coefficient

$a_{k_1k_2}(t,x)$ of the derivative $\partial^{k_1+k_2}/\partial t^{k_1}\partial x^{k_2}$ of the operator L expands into the asymptotic series

$$a_{k_1k_2} = \sum_{m=2k_1p+k_2-2lp}^{m=\infty} t^{m/2p} a_{k_1k_2}^{(m)}\left(\frac{x}{t^{1/2p}}\right), \tag{4}$$

where

$$a_{k_1k_2}^{(2k_1p+k_2-2lp)} \equiv 0 \quad \text{for } 2k_1p + k_2 < 2lp.$$

The coefficient $b_{k_1k_2}^{j}$ at the derivative $\partial^{k_1+k_2}/\partial t^{k_1}\partial x^{k_2}$ of the operator B_j is expanded in the asymptotic series

$$b_{k_1k_2}^{(j)} = \sum_{m=2k_1p+k_2-m_j}^{\infty} t^{m/2p} b_{k_1k_2}^{jm}\left(xt^{-1/2p}\right) \tag{5}$$

and $b_{k_1k_2}^{jm} = 0$ if $2k_1p + k_2 < m_j$; $\quad m = 2k_1p + k_2 - m_j$.

The functions $a_{k_1k_2}^{m}$, $b_{k_1k_2}^{jm}$ are infinitely differentiable. The operators obtained from L_0, B_{j0}—the principal parts of the operators L, B_j—by replacing $a_{k_1k_2}$, $b_{k_1k_2}^{j}$ by $a_{k_1k_2}^{0}$, $b_{k_1k_2}^{j0}$, respectively, are denoted by $L_0(0)$, $B_{j0}(0)$.

Expansions analogous to (4), (5) are also required in a neighborhood of the second distinguished point of the boundary. At all non-distinguished points of the boundary the operators L, B_j must satisfy a condition analogous to the Shapiro–Lopatinskii condition for elliptic problems. One more condition is imposed at the distinguished point.

It is required that the special problem

$$L_0(0)u = f,$$

$$B_{j0}(0)u = \varphi_j \tag{6}$$

satisfy a condition of Shapiro–Lopatinskii type on the entire surface Γ_0:

$$t^q = \sum_{i_1+\cdots+i_n=2pq} a_{i_1\cdots i_n} x_1^{i_1}\cdots x_n^{i_n}, \tag{7}$$

except for the point $(0,0)$.

The study of problem (1), (3) begins with the investigation of problem (6) in the domain G_0, bounded by the paraboloid (7). Define the space $\dot{H}_{\alpha}^{k\,2p}(G_0)$, for integer k, as the space with norm

$$\|u\|^2_{\dot{H}_{\alpha}^{k\,2p}(G_0)} = \sum_{2pi_1+i_2\leq 2pk} \iint_{G_0} t^{\alpha-(2kp-2i_1p-2i_2)/p} \left| \frac{\partial^{i_1+i_2} u}{\partial t^{i_1} \partial x^{i_2}} \right|^2 dt\, dx.$$

An important role for us is played by the transformation

$$t = e^{-2p\tau}, \qquad x_i = \omega_i e^{-\tau}, \tag{8}$$

which maps the domain G_0 into a cylinder D_0, the equation of whose boundary D_0' has the form

$$1 = \sum a_{i_1 \dots i_n} \omega_1^{i_1} \cdots \omega_n^{i_n}.$$

It is easy to verify that a function u from $\dot{H}_{\alpha}^{k\,2p}(G_0)$, after the transformation (8), is transformed into a function u_1 such that $u_1 e^{(-n-2p-2\alpha p+4kp)/2\tau}$ belongs to the Slobodetskii space $H_2^{k\,2p}(D_0)$. For arbitrary k the norm of u in $\dot{H}_{\alpha}^{k\,2p}(G_0)$ is defined to be equal to the norm of the function $u_1 e^{(-n-2p-2\alpha p+4kp)/2\tau}$ in $H_2^{k\,2pk}(D)$. The space of boundary functions is introduced analogously: $\dot{H}_{\alpha}^{k\,2p}(\Gamma_0)$. After the transformation (8) and the subsequent Fourier transform in τ, problem (6) passes into a certain problem:

$$\widetilde{L}_0(\omega, \lambda, \partial/\partial\omega)\tilde{u} = \widetilde{F}(\omega, \lambda),$$

$$\widetilde{B}_{j0}(\omega, \lambda, \partial/\partial\omega)\tilde{u} = \widetilde{\Phi}_j(\omega, \lambda) \tag{9}$$

in an infinitely smooth domain of the space $(\omega_1, \dots, \omega_n)$. Using the results of [2], we obtain that there exists an operator $R(\lambda)$ such that $\tilde{u} = R(\lambda)[\widetilde{F}, \widetilde{\Phi}_j]$, where $R(\lambda)$ is a meromorphic function of λ and in each

in the strip $|\operatorname{Im}\lambda| <$ const it has a finite number of poles. With the help of $R(\lambda)$ one can solve problem (1), (3) in the domain (2). This problem turns out to be solvable if on the line

$$\operatorname{Im}\lambda = \frac{-n-2p-2\alpha p+4kp+4lp}{2}$$

$R(\lambda)$ is regular. Further, using the Fourier transform, one solves problem (6) in the half-space $x_n > 0$ and in the whole space (t, x).

After this we solve problem (1), (3) in a bounded domain by constructing a regularizer. For this purpose the surface (2) is reduced in a neighborhood of the point $(0,0)$ to the form (7) by the transformation

$$t = t',$$
$$x = x_i' + \sum_{s>1} t^{s/2p}\psi_{si}\left(\frac{x}{t^{1/2p}}\right).$$

This transformation preserves the form of equations (1), (3). In some cases (for example, when $p = 1$ or $n = 1$) this transformation turns out to be infinitely differentiable. The regularizer can be constructed according to the schemes of work ([2]). For a solution of problem (1), (3) belonging to $\dot{H}_{\alpha}^{k_1+2p}$, under the condition

$$f \in \dot{H}_{\alpha_1}^{k,2p}(G), \qquad \varphi_j \in \dot{H}_{\alpha_1}^{k+l-m_j/2p-1/4p,\,2p}(\Gamma)$$

one can obtain, in a neighborhood of the point $(0,0)$, the representation

$$u = \sum_{0<s<s_j} a_{jsq} t^{-i\lambda_j/2p} \ln^s t\, P_{sjq}(t \ln^q t) + w, \tag{10}$$

$$\frac{-n-2p-2\alpha p+4k_1p+4lp}{2} < \operatorname{Im}\lambda_j < \frac{-n-2p-2\alpha_1 p+4kp+4lp}{2},$$

where λ_j are poles of $R(\lambda)$ of multiplicity s_j; P_{sjq} is a polynomial with coefficients—infinitely differentiable functions of $x/t^{1/2p}$; a_{jsq} are constants depending on u; and w is a function from the space $\dot{H}_{\alpha_1}^{k+l,2p}(G)$. This representation can be used to study the smoothness of the solution.

The case of the Slobodetskii spaces $H_2^{k,2pk}(G)$ is included in ours in the following way. If the function $v(x,t)$ is such that

$$\iint_G |v|^2\,dx\,dt + \iint_G t^{\alpha}\left|\frac{\partial^{k_1+k_2}v}{\partial t^{k_1}\partial x^{k_2}}\right|^2 dt\,dx < \infty,$$

$$2pk_1 + k_2 = 2pk; \quad \text{the number } -n/2 - p - 2\alpha p + 4k \text{ is not an integer},$$

then one can prove that the function v has the form

$$v = p(x,t) + v_1,$$

where $p(x,t)$ is a polynomial whose generalized order is $(2p-1)$, and $v_1 \in \dot{H}_{\alpha}^{k,2p}(G)$. After making the corresponding change of variables, one can pass to the spaces $\dot{H}_{\alpha}^{k,2p}(G)$.

By investigating the location of the poles of the function $R(\lambda)$, one can obtain, in concrete cases, various smoothness theorems.

In the case when problem (1), (3) is a parabolic semibounded problem (for the definition see work ([2])), the poles of $R(\lambda)$ are situated below some line $\operatorname{Im}\lambda = K$.

Using this, one can show that a sufficiently smooth solution of problem (1), (3) is infinitely differentiable. If the boundary conditions are Dirichlet conditions, and equation (1) has the form

$$\frac{\partial u}{\partial t} = (-1)^p \sum_{i=0}^{2p} a^{(i)}_{i_1\cdots i_n} \frac{\partial^{2p-i} u}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} + f = Lu + f, \tag{11}$$

$$\operatorname{Re} \sum a^{(0)}_{i_1\cdots i_n} (i\lambda_1)^{i_1} \cdots (i\lambda_n)^{i_n} > 0, \qquad f \in H_2^{k,2pk}(G), \quad \text{then } u \in H_2^{k+1,\,2pk+2p}(G),$$

In this case it is shown that the function $R(\lambda)$ has no poles in the upper complex half-plane. In the case of the analogous problem for the equation

$$\frac{\partial u}{\partial t} = -(-1)^p \sum_{i=0}^{2p} a^{(i)}_{i_1\cdots i_n} \frac{\partial^{2p-i}}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} + f$$

the smoothness of the solution depends on the numbers $a_{i_1\cdots i_n}$ entering into the equation of the boundary surface. It can be proved that the smoothness improves when the quantity

$$\max \left(\sum_{i_1+\cdots+i_n=2pq} a_{i_1\cdots i_n} \lambda_1^{i_1} \cdots \lambda_p^{i_n} \Big/ \left(\lambda_1^{2pq} + \cdots + \lambda_n^{2pq}\right) \right)$$

is decreased.

In the case when the boundary conditions are Dirichlet conditions and f is a function from $L_2(G)$, the problem was studied by V. P. Mikhailov in paper [1]. The case when the domain contains no singular points was studied in paper [2].

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Note: Figure translations are in progress. See original paper for figures.