MATHEMATICS

1965

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Abstract

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MATHEMATICS

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STABILITY OF DIFFERENCE SCHEMES FOR THE CAUCHY PROBLEM FOR SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

(Presented by Academician S. L. Sobolev, February 22, 1965)

It is known that from the stability of a difference scheme approximating the Cauchy problem for linear systems of partial differential equations, under very broad assumptions, there follows the convergence of the solutions of the difference equations to the solution of the Cauchy problem and the correctness of this problem ([1]). For separate classes of correct systems (hyperbolic, parabolic), stable difference schemes for the Cauchy problem have been constructed. The converse question—whether stable difference schemes exist for every correct Cauchy problem—is apparently difficult, since in the general case there is no detailed investigation of the problem even for the differential equations themselves. However, for linear systems with constant coefficients the question is resolved relatively simply. Here it will be shown that for such systems there always exists a stable explicit difference scheme for the Cauchy problem, if this problem is correct. For these same systems there exist implicit unconditionally stable difference schemes.

Consider a system correct in the sense of Petrovsky ([2,3]):

$$\partial u/\partial t = P(\partial/\partial x)u \tag{1}$$

with initial condition

$$u(x,0) = u_0(x). \tag{2}$$

Here $x = (x_1, x_2, \dots, x_N)$; $u = (u_1, u_2, \dots, u_m)$; $P(\lambda)$ is a matrix whose elements are polynomials in $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ of degree not higher than p. An example of such a system may be a system describing the propagation of sound and heat in a one-dimensional flow of a compressible fluid. This system is neither parabolic nor hyperbolic. Some difference schemes for it are considered in ([4]).

For simplicity we shall assume that the initial data are periodic in each of the variables. The estimates will be given in integral norms. With the help of embedding theorems ([5]), the results can also be reformulated in C_k norms. By

$|u|_0$ we shall denote the norm of the vector-function $u(x)$ in the space $L_2(D)$, and by $|u|_r$ the norm in the space $W_2^r(D)$ ([5]), where the domain D is the parallelepiped $(0, l_1) \times (0, l_2) \times \cdots \times (0, l_N)$; l_j is the period of the function with respect to the variable x_j.

Replace the differential system (1) by a difference system approximating it,

$$(u_{n+1} - u_n)/\tau = P(\Delta) u_n, \tag{3}$$

where $\tau = t/n$; $\Delta = (\Delta_1, \Delta_2, \dots, \Delta_N)$; the operator

$$\Delta_j u = [u(x_1, x_2, \dots, x_j + h, \dots, x_N) - u(x_1, x_2, \dots, x_j - h, \dots, x_N)]/2h.$$

Theorem 1. Suppose that for the solutions of problem (1), (2) the estimate

$$|u(x,t)|_0 \le c_1 |u_0(x)|_r \tag{4}$$

for some integer r and $0 \leqslant t \leqslant T$, where the constant c_1 depends only on T. Then the difference scheme (3) is stable for $u_0(x) \in W_2^r$, if

$$\tau < c_2 h^{2p}. \tag{5}$$

For the proof we shall apply, as usual ([4]), the Fourier method. Without loss of generality one may assume that the period of the initial function in each of the variables is equal to 2π. Represent the solution of the difference system in the form

$$u_n(x) = \sum_{|k|=0}^{\infty} \gamma_{n,k} e^{ikx}, \tag{6}$$

where $k = (k_1, k_2, \dots, k_N)$ are integer-valued vectors; $\gamma_{n,k}$ are m-dimensional vectors. Then

$$|u_0(x)|_r^2 \sim \sum_{|k|=0}^{\infty} |\gamma_{0,k}|^2 (1 + |k|^r)^2,$$

while

$$|u(x,t)|_0^2 \sim \sum_{|k|=0}^{\infty} |\gamma_{n,k}|^2,$$

and, in order to verify stability, it is sufficient to establish that

$$|\gamma_{n,k}| \leqslant c_3 |\gamma_{0,k}| (1 + |k|^r).$$

Here and below the c_j denote positive constants independent of the initial function $u_0(x)$ and of the quantities n, τ, h, and k.

It follows from equations (3) that

$$\gamma_{1,k} = \gamma_{0,k} + \tau(P(\Delta)\gamma_{0,k}e^{ikx})e^{-ikx} = (E + \tau P(is))\gamma_{0,k},$$

and, consequently,

$$\gamma_{n,k} = (E + \tau P(is))^n \gamma_{0,k},$$

where

$$s = (\sin k_1 h/h, \sin k_2 h/h, \dots, \sin k_{Nh}/h).$$

On the other hand, condition (4), applied to the function γe^{isx}, entails the inequality

$$\|\exp\{P(is)t\}\| \leqslant c_4(1 + |s|^r)$$

for all real s. Thus it is sufficient to prove boundedness of the norm

$$\|(E + \tau P(is))^n \exp\{-P(is)t\}\|$$

uniformly with respect to all real s and τ for $0 \leqslant n\tau = t \leqslant T$. Since

$$\|P(is)\| \leqslant c_5 h^{-p}, \tag{7}$$

it follows, in view of condition (5), that for sufficiently small h the norm $\|\tau P(is)\| < 1/2$, and

$$(E + \tau P(is))^n = \exp\{n \ln(E + \tau P(is))\}.$$

Then

$$\begin{aligned} A &= (E + \tau P(is))^n \exp\{-P(is)t\} \\ &= \exp\{n \ln(E + \tau P(is)) - P(is)t\} \\ &= \exp\{n[\tau P(is) - \tfrac{1}{2}\tau^2[P(is)]^2(1 + O(\tau\|P(is)\|))] - P(is)t\} \\ &= \exp\{-n\tau \cdot \tfrac{1}{2}\tau[P(is)]^2(1 + O(\tau\|P(is)\|))\}. \end{aligned}$$

From inequalities (5) and (7) the boundedness of the norm of the matrix A follows. The theorem is proved.

Condition (5) on the time step is, of course, very burdensome. For parabolic systems, for example, the condition $\tau < c_2 h^p$ is sufficient. But in the general case, as is seen from the simplest examples, condition (5) cannot be weakened for scheme (3).

Correct systems for which inequality (4) is satisfied with $r = 0$ were considered in ([6]). For these systems stable difference schemes were constructed under the condition $\tau < c_2 h^p$. It is probably possible to construct schemes with such a condition also for general correct systems with constant coefficients; however, this condition too imposes a very strong restriction on the time step. It is therefore of interest to study difference schemes that are stable under arbitrary relations between h and τ, i.e., unconditionally stable schemes.

Theorem 2. The difference scheme

$$(u_{n+1} - u_n)/\tau = P(\Delta)u_{n+1} \tag{8}$$

for system (1), correct in the sense of Petrovskii, is unconditionally stable if the initial function $u_0(x) \in W_2^r$, where $r = p(m-1)$.

Proof. We apply, as above, the Fourier method, representing the solution in the form (6). Then

$$\gamma_{n,k} = (E - \tau P(is))^{-n}\gamma_{0,k},$$

where

$$s = (\sin k_1 h/h, \sin k_2 h/h, \ldots, \sin k_{Nh}/h).$$

In order to prove the theorem, it is sufficient to verify the validity of the estimate

$$\|(E - \tau P(is))^{-n}\| \leqslant c_1(1 + |k|^r). \tag{9}$$

for all sufficiently small τ when $0 < n\tau = t \leq T$. If the function of the matrix is represented by the interpolation polynomial (7), then one can obtain the estimate

$$\|f(A)\| \leq c_2 \sum_{l=0}^{m-1} \|A\|^l \max_{z\in D} |f^{(l)}(z)|, \tag{10}$$

where D is the smallest convex polygon in the complex z-plane containing all eigenvalues of the matrix A (3). Putting in the inequality (10) the matrix $A = \tau P(is)$, and the function $f(\lambda) = (1-\lambda)^{-n}$, we obtain

$$\|(E - \tau P(is))^{-n}\| \leq c_2 \sum_{l=0}^{m-1} (\tau\|P(is)\|)^l \max_{z\in D} \left|\frac{n(n+1)\cdots(n+l-1)}{(1-z)^{n+l}}\right|. \tag{11}$$

For systems that are correct in the sense of Petrovskii, the real parts of the eigenvalues of the matrix $P(is)$ are bounded for all real s (3). Consequently, $\max_{z\in D} |1-z|^{-n-l} < c_3$ for all sufficiently small τ. If one takes into account that $\|P(is)\| \leq c_4(1 + |k|^p)$, while $\tau^l n(n+1)\cdots(n+l-1) < c_5$, then the estimate (9) follows from (11). The theorem is proved.

A major drawback of implicit systems in the case of several spatial variables is the need to invert, at each step, a matrix of high order. With the aid of the fractional-step method or the splitting method, in a number of cases this problem can be substantially simplified [8–11]. The splitting method is sometimes also applicable to the systems under consideration. It is necessary to note that in paper [9] a very general theorem is proved on the applicability of the splitting method to a system of differential equations. However, in [9] the stability requirement is actually included in the hypothesis of the theorem. It therefore seems expedient to formulate a more special, but more concrete, assertion.

Suppose that in the equations of system (1) there are no mixed derivatives, i.e.

$$P\left(\frac{\partial}{\partial x}\right) = \sum_{j=1}^{N} P_j\left(\frac{\partial}{\partial x_j}\right). \tag{12}$$

Without loss of generality one may assume that $P_j(0) = 0$. Consider the difference scheme

$$(u_{n+j/N} - u_{n+(j-1)/N})/\tau = P_j(\Delta_j) u_{n+j/N}, \tag{13}$$

where $j = 1, 2, \dots, N;\ n = 0, 1, 2, \dots;\ n\tau = t \leq T$.

Theorem 3. Suppose that for system (1), correct in the sense of Petrovskii, relation (12) holds, where the matrices $P_j(is_j)$ commute with one another for different $j = 1, 2, \dots, N$ and all real s_j. Then the difference scheme (13) approximates problem (1), (2) and is unconditionally stable for an initial function $u_0(x) \in W_2^r$, where $r = pN(m-1)$.

The proof of approximation of problem (1), (2) by scheme (13) does not differ in essence from the proof carried out in [8] for the heat equation (see also [9]). In order to prove stability, note that the eigenvalues of each of the matrices $P_j(is_j)$ have bounded real parts, since $P_j(is_j) = P(0, 0, \dots, is_j, \dots, 0)$. If again the solution is represented in the form (6), then for the difference scheme (13)

$$\gamma_{n,k} = \left\{ \prod_{j=1}^{N} \left(E - \tau P_j \left(i \frac{\sin k_j h}{h} \right) \right)^{-1} \right\}^n \gamma_{0,k}.$$

In view of the commutativity of the matrices $P_j(is_j)$, it is sufficient to establish the estimate

$$\left\| \left(E - \tau P_j \left(i \frac{\sin k_{jh}}{h} \right) \right)^{-n} \right\| \leq c_1 (1 + |k_j|^{p(m-1)}),$$

which is verified as in the preceding theorem.

The commutativity condition, of course, sharply reduces the class of systems and is in practice seldom satisfied, unless system (1) decomposes into separate equations. However, this condition cannot be discarded, as is illustrated by the simplest system
$\partial u_1/\partial t = \partial^2 u_2/\partial x_1^2;\ \partial u_2/\partial t = -\partial^2 u_1/\partial x_2^2$. One can verify that difference scheme (13) is not unconditionally stable for this system. The splitting method

for this system does not lead to an unconditionally stable scheme, no matter how the operator $\partial^2 u_2/\partial x_1^2$ is approximated by the difference operator $Q_1 u_2$ with respect to the variable x_1, and the operator $\partial^2 u_1/\partial x_2^2$ by the difference operator $Q_2 u_1$ with respect to the variable x_2.

Remark 1. Instead of the difference scheme (8), one may consider the scheme $(u_{n+1} - u_n)/\tau = aP(\Delta)u_{n+1} + (1-a)P(\Delta)u_n$. If $1/2 \leq a \leq 1$, then the conclusion of Theorem 2 is also valid for this scheme. The proof is almost unchanged. An analogous remark is applicable also to Theorem 3.

Remark 2. Instead of periodic initial data one may consider functions that decrease sufficiently rapidly at infinity. In the proof, the Fourier series is then replaced by the Fourier integral. However, the actual computational aspect of the question remains unclear in this case (at least for implicit schemes): the need arises to invert infinite matrices. Whether, for an arbitrary correct system (1), these matrices can be approximately replaced by finite ones, and how this should be done, is unknown.

3. The theorems proved above also apply to certain special cases of boundary-value problems. Suppose, for example, that the domain is a parallelepiped with edges parallel to the coordinate axes, and that in system (1) only even derivatives with respect to each of the spatial variables occur. Suppose that the boundary conditions consist in the vanishing of the function and of several even derivatives in the normal direction. Then, extending the initial function oddly across the boundary, one can reduce the boundary-value problem to a Cauchy problem with periodic initial conditions.

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Received
13 II 1965

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Note: Figure translations are in progress. See original paper for figures.