ON A PROPERTY OF “INSTABILITY” OF HARMONIC CAPACITY

MATHEMATICS

1965

SovietRxiv

Abstract

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UDC 517.947.42

MATHEMATICS

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ON A PROPERTY OF "INSTABILITY" OF HARMONIC CAPACITY

(Presented by Academician M. A. Lavrent' ev, 9 IV 1965)

1. Let e be a Borel subset of three-dimensional Euclidean space C^3. We shall use the following notation: $\gamma(e)$ is the capacity (relative to the Newtonian potential; see, for example, ([1])) of the set e; $\bar{e}$ is the closure, ∂e the boundary of the set e. The open ball of radius δ with center at the point P will be denoted by $K(P,\delta)$. We note that $\gamma[K(P,\delta)] = \delta$.

The present note is adjacent to the author' s works ([3], [4]); see also there for a discussion of the results.

Let D be an open subset of the space C^3. In ([4]) the following property of "instability" of capacity was proved (cf. ([4]), Theorem 2):

Let

1°. For almost all points $P \in C^3$ (or, what is the same, for almost all P belonging to the complement of D),

$$\overline{\lim_{\delta\to 0}} \frac{\gamma[D\cap K(P,\delta)]}{\delta^3} = \infty.$$

Then

2°. For all $P \in C^3$ and $\delta > 0$,

$$\gamma[D\cap K(P,\delta)] \geq A_0\delta,$$

where $A_0 > 0$ is an absolute constant.

With the aid of a theorem of Keldysh–Brelot it is not difficult to show (this will be done in § 2) that from property 2° there follows the following property:

3°. For all $P \in C^3$ and $\delta > 0$,

$$\gamma[D\cap K(P,\delta)] = \gamma[K(P,\delta)] = \delta.$$

Thus, the following is true.

Theorem 1. *If for almost all (with respect to Lebesgue measure in C^3) points $P \in C^3$*

$$\overline{\lim_{\delta\to 0}} \frac{\gamma[D \cap K(P,\delta)]}{\delta^3} = \infty,$$

then for all $P \in C^3$ and $\delta > 0$

$$\gamma[D \cap K(P,\delta)] = \delta.$$

Consequently, conditions 1° and 3° are equivalent.

Theorem 1 makes it possible to formulate a necessary and sufficient condition for the possibility of uniform approximation of continuous functions by harmonic functions in the following form (cf. ([4]), Theorems A and B). Let $E \subset C^3$ be a compact set; $C(E)$ the space of all functions continuous on E, $f(P)$, $P \in E$, with norm $\|f\| = \max_{P\in E} |f(P)|$; $H(E)$ the subspace of $C(E)$ consisting of all functions admitting uniform approximation on E by functions harmonic on E. Applying Theorem 1 to the complement $\mathcal{D}$ of the compact set E and Theorem B of ([4]), we obtain:

Theorem 2. *In order that $C(E) = H(E)$, it is necessary and sufficient that for all $P \in C^3$ and $\delta > 0$*

$$\gamma[K(P,\delta) \setminus E] = \delta.$$

Theorem 1 can be strengthened somewhat.

Theorem 3. *If property 1° is satisfied, then for any bounded open set $G \subset C^3$*

$$\gamma(D \cap G) = \gamma(G).$$

It follows from this that Theorem 2 also remains valid if, instead of $K(P,\delta)$, one takes an arbitrary open set G. More precisely, the following holds.

Theorem 4. *In order that $C(E) = H(E)$, it is necessary and sufficient that, for every bounded open set G,*

$$\gamma(G \setminus E) = \gamma(G).$$

2. Let F be a closed set; let g_n, $n = 1, 2, \dots$, $F \subset \overline{g}_{n+1} \subset g_n$, be a sequence of open sets converging to F in the sense that each point of the complement of F belongs to only a finite number of the sets g_n. Suppose that each of the sets g_n is bounded by a finite number of smooth closed Jordan surfaces. Let $\varphi(P)$ be an arbitrary function, defined and continuous on

the boundary ∂F of the set F; let $\widetilde{\varphi}(P)$ be its continuous extension to the whole space C^3; let $h_n(P)$ be the solution of the Dirichlet problem in g_n with boundary data $\varphi(P)$, $P \in \partial g_n$ (if g_n contains the exterior of some ball, then $h_n(P) \to 0$ as $\overline{OP} \to \infty$). The following theorem was proved by M. V. Keldysh ([7]) for closed domains and by M. Brelot ([2]) for arbitrary closed sets:

If each point $P \in \partial F$ is a regular point for the complement of F, i.e.

$$\int_0 \frac{\gamma[K(P,\delta) \setminus F]}{\delta^2}\, d\delta = \infty,$$

then the sequence $h_n(P)$, $n = 1, 2, \dots$, converges uniformly on ∂F to the function $\varphi(P)$.

From the Keldysh–Brelot theorem there follows

Lemma. *Let g be a bounded open set. If each point $P \in \partial g$ is a regular point of the set g, i.e.*

$$\int_0 \frac{\gamma[g \cap K(P,\delta)]}{\delta^2}\, d\delta = \infty, \tag{1}$$

then $\gamma(g) = \gamma(\overline{g})$.

Indeed, let F be the complement of g; let $W(P)$ be the potential of the set $\overline{g}$. From the regularity condition for the points $P \in \partial g$ it follows that $W(P) = 1$ for $P \in \overline{g}$, in particular, $W(P) = 1$ on the boundary of the set F. Put $\widetilde{\varphi}(P) = W(P)$. Let g_n be a sequence of "good" open sets converging to F (see above); let e_n be the complement of g_n. The solution of the Dirichlet problem in g_n with boundary data $\widetilde{\varphi}(P) = 1$, $P \in \partial g_n$, is the potential $W_n(P)$ of the set e_n. From the Keldysh–Brelot theorem it follows that $W_n(P)$ converges uniformly to $W(P)$ on ∂F, and consequently also on F. Hence it follows that $\gamma(e_n) \to \gamma(\overline{g})$; since $e_n \subset g$, we obtain the assertion of the lemma.

We now show that $2^\circ \Rightarrow 3^\circ$. Fix a point $P_0 \in C^3$ and $\delta_0 > 0$. Put $g = D \cap K(P_0, \delta_0)$. Obviously, we have $\overline{g} = \overline{K(P_0, \delta_0)}$ (from 2° it follows that D is an everywhere dense set); hence $\gamma(\overline{g}) = \delta_0$. We show that at every point $P \in K(P_0, \delta_0)$ condition (1) is satisfied. At every point $P \in K(P_0, \delta_0)$ condition (1) is obviously satisfied, since for sufficiently small $\delta > 0$

$$\gamma[g \cap K(P,\delta)] = \gamma[D \cap K(P,\delta)] \geq A_0 \delta.$$

Let P be a boundary point of the ball $K(P_0, \delta_0)$. Consider the ball $K(P, \delta)$; for sufficiently small $\delta > 0$, $K(P, \delta)$ contains a ball $K(P', \delta/4)$ which belongs entirely to $K(P_0, \delta_0)$. We have

$$\gamma[g \cap K(P,\delta)] \geq \gamma\left[g \cap K\left(P', \frac{\delta}{4}\right)\right] = \gamma\left[D \cap K\left(P', \frac{\delta}{4}\right)\right] \geq \frac{A_0}{4}\delta,$$

whence (1) follows. Thus (1) holds for all $P \in \partial g \in \overline{K}(P_0, \delta_0)$. By the lemma,

$$\gamma(g) = \gamma(\overline{g}) = \delta_0.$$

Property 3^0 is proved.

In a completely analogous way one can show that if property 2^0 is satisfied, then for open sets Γ with a sufficiently good boundary

$$\gamma(D \cap \Gamma) = \gamma(\Gamma) = \gamma(\overline{\Gamma}).$$

This implies Theorem 3.

3. Analogues of the results obtained above are also valid for m-dimensional Euclidean space C^m (we retain the notation adopted above for the case $m = 3$). The capacity γ (with respect to the Newtonian potential in C^m) of the ball $K(P,\delta)$ of radius δ in the space C^m is equal to δ^{m-2}. Accordingly, analogues of Theorems 1 and 2 in m-dimensional space are formulated as follows*.

Theorem 1′. If for almost all (with respect to Lebesgue measure in C^m) points $P \in C^m$

$$\overline{\lim_{\delta\to 0}} \frac{\gamma[D \cap K(P,\delta)]}{\delta^m} = \infty,$$

then for all $P \in C^m$ and $\delta > 0$

$$\gamma[D \cap K(P,\delta)] = \delta^{m-2}.$$

Theorem 2′. In order that $C(E) = H(E)$, $E \subset C^m$, it is necessary and sufficient that for all $P \in C^m$ and $\delta > 0$

$$\gamma[K(P,\delta) \setminus E] = \delta^{m-2}.$$

Theorem 4 for $E \subset C^m$ is valid in the same formulation.

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Received
11 III 1965

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* We note that in works (3, 4), in the formulation of the approximation theorem for the case C^m there is an inaccuracy; the inequality for the capacity in Theorem B′, b) should be as follows: $\gamma(P, \delta) \geq A_0 \delta^{m-2}$.

Note: Figure translations are in progress. See original paper for figures.