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Abstract

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THE CAUCHY PROBLEM AND A BOUNDARY-VALUE PROBLEM FOR SOME NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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Consider the following boundary-value problems:

$$y'' + \frac{2}{x}y' - y + y^n = 0, \qquad n > 1, \qquad x \geq 0; \tag{1}$$

$$y(0) = y_0 < \infty, \qquad y'(0) = 0, \qquad y(\infty) = 0, \tag{2}$$

y_0 —an unknown positive parameter;

$$\eta'' = \eta - \frac{\eta^n}{x^{n-1}}, \qquad n > 1, \qquad x \geq 0; \tag{3}$$

$$\eta(0) = 0, \qquad \eta'(0) = \alpha < \infty, \qquad \eta(\infty) = 0, \tag{4}$$

α —an unknown positive parameter;

$$z'' + \frac{2}{x}z' - z + |z|^{n-1}z = 0, \qquad n > 1, \qquad x \geq 0; \tag{5}$$

$$z(0) = z_0 < \infty, \qquad z'(0) = 0, \qquad z(\infty) = 0, \tag{6}$$

z_0 —an unknown positive parameter.

Equation (3) is obtained from (1) by the substitution $\eta(x) = x \cdot y(x)$. If problem (3)–(4) is solvable, then problem (1)–(2) is also solvable, with $y_0 = \alpha$. Problem (1)–(2) arises, in particular, in finding spherically symmetric particle-like solutions of equations for a complex scalar field of the form

$$\nabla^2\psi - \frac{\partial^2\psi}{\partial x_0^2} - m^2[1 + F'(\psi^*\psi)]\psi = 0,$$

if

$$F(z) = -\frac{2\lambda}{n+1}\, z^{(n+1)/2}, \qquad F'(z) = \frac{dF(z)}{dz}, \qquad \psi = u(r)e^{-i\varepsilon x_0}, \quad \varepsilon < m$$

and

$$x = r\sqrt{m^2 - \varepsilon^2}, \qquad u = \left(\frac{m^2 - \varepsilon^2}{\lambda m^2}\right)^{1/(n-1)} y.$$

For $n = 3/2$ these boundary-value problems occur in the statistical theory of the nucleus. The investigation of the solvability of problems (1)–(6) and of the properties of their solutions is an essential part of papers ([1]). However, only by the results of papers ([2,3]) has the existence of positive solutions of these problems been rigorously proved for every real n such that $1 < n \leq 4$. Nehari ([2]) showed the unsolvability of problems (1)–(6) for $n = 5$. The present note is devoted to the question of the existence of nonpositive solutions of the indicated problems.

In the case $n = (2p+1)/2q$ (p and q are natural numbers), any solution of equations (1), (3) has at most one zero on the interval $0 < x < \infty$ and is not continuable beyond this zero. For such n, problems (1)–(2) and (3)–(4) cannot have any solutions except positive ones.

In the case $n = 2p$ or $n = 2p/(2q+1)$, the solutions of equations (1) and (3) cannot have negative minima, and all solutions of the corresponding boundary-value problems are positive for $0 < x < \infty$, if these solutions exist. It remains to consider the case $n = (2p+1)/(2q+1)$. Since equations (1), (3), (5) have a singularity at the point $x = 0$, we shall state separately a general result for their solutions.

Lemma 1. *For any $a > 0$ and any $n > 1$, $n = (2p+1)/(2q+1)$ (p and q are natural numbers), there exists a unique solution of the Cauchy problem for each of equations (1) and (3) under the initial conditions $y(0) = a$, $y'(0) = 0$, $\eta(0) = 0$, $\eta'(0) = a$, defined for all $x > 0$. Equation (5), under the conditions $z(0) = a$, $z'(0) = 0$, has a unique solution, defined for all $x > 0$, for any real $n > 1$. These solutions of equations (1), (3), (5), together with their first derivatives with respect to x, depend continuously on a on any finite interval $0 \leq x \leq X < \infty$.*

Lemma 1 is a consequence of analogous lemmas from the work ([3]) of E. P. Zhidkov and the author of the present note.

The following theorem proves the existence of a countable set of solutions of problems (1)–(6) for certain values of n.

Theorem 1. *For any nonnegative integer i ($i = 0, 1, 2, \ldots$) and any $n = (2p+1)/(2q+1)$ (p and q are natural numbers), $1 < n < 4$, there exist solutions $y = y_i(x)$, $\eta = \eta_i(x)$ of problems (1)–(2) and (3)–(4), having exactly i zeros on the interval $0 < x < \infty$.*

Problem (5)–(6) has a solution $z = z_i(x)$ with i zeros on the interval $0 < x < \infty$ for any real $n > 1$, $n < 4$.

The proof of the theorem uses the properties of solutions of the Cauchy problems for equations (1), (3), and (5), established by Lemma 1 and by the following two auxiliary lemmas.

Lemma 2. *For any nonnegative integer i ($i = 0, 1, 2, \ldots$) and any $n = (2p+1)/(2q+1)$, $1 < n < 4$, there exist solutions $y = y(x)$ and $\eta = \eta(x)$ of equations (1) and (3), $y(0) > 0$ and $y'(0) = 0$, $\eta(0) = 0$ and $\eta'(0) > 0$, having at least $i+1$ zeros on the interval $0 < x < \infty$. For equation (5) there exists a solution $z = z(x)$, $z(0) > 0$ and $z'(0) = 0$, having at least $i+1$ zeros on this interval for any real $n > 1$, $n < 4$.*

For the proof, a change of variables is made in equation (3) according to the formulas: $\eta(x) = e^x v(t)$, $t = \frac{1}{2}(1 - e^{-2x})$, $t < \frac{1}{2}$. For the equation $v'' + v^n/f(t) = 0$, where $v'' = d^2v/dt^2$ and $f(t) = [-\frac{1}{2}\ln(1-2t)]^{n-1}(1-2t)^{(n+3)/2}$, on some interval $0 < t < b < \frac{1}{2}$ there exists a solution $v = v(t)$, $v(0) = v(b) = 0$, having i zeros in this interval. This solution is found as the function realizing the minimum of the functional

$$I(w) = \int_0^b \left[{w'}^2 - \frac{2}{n+1}\,\frac{w^{n+1}}{f(t)} \right] dt$$

in the class of continuous and piecewise continuously differentiable functions w on $[0, b]$, $w(0) = w(b) = 0$, having i zeros inside this interval. The functions w are also required to satisfy the normalization condition:

$$\int_0^b {w'}^2\, dt = \int_0^b \frac{w^{n+1}}{f(t)}\, dt.$$

Lemma 3. *Let $y = y(x)$, $\eta = \eta(x)$, $z = z(x)$ be solutions of equations (1), (3), and (5) such that $y(x_0) = 0$, $\eta(x_0) = 0$, $z(x_0) = 0$ and $y'(x_0) = \varepsilon_1$, $\eta'(x_0) = \varepsilon_2$, $z'(x_0) = \varepsilon_3$ at some point $x_0 > 0$. For sufficiently small $|\varepsilon_i|$, $i = 1, 2, 3$, and sufficiently large x_0, the functions $y(x)$, $\eta(x)$, $z(x)$ have no zeros on the half-line $x > x_0$.*

The proof of the lemma is carried out by estimating the rate of decrease of the "energy" E after the moment $x = x_0$. For $\eta = \eta(x)$, the function E is defined by the equality

$$E = \frac{1}{2}\eta {a'}^2(x) - \frac{1}{2}\eta^2(x) + \frac{1}{n+1}\frac{\eta^{n+1}(x)}{x^{n-1}}.$$

We now outline the method of proof of Theorem 1.
The solution $y = y_i(x)$ of problem (1)–(2) is found on the boundary of the family of solutions $y = y(x)$ of equation (1), with $y(0) > 0$ and $y'(0) = 0$, having at least $i+1$ zeros for $x > 0$, and the family of solutions of this equation having exactly i zeros for $x > 0$. The solutions of the second family, by Lemma 3, are, in a neighborhood, $y = y_{i-1}(x)$, $0 < x < T_{i-1} < \infty$.

The solutions $\eta = \eta_i(x)$ and $z = z_i(x)$ of problems (3)–(4) and (5)–(6) are found analogously. We note here an important property of the solutions of the boundary-value problems under consideration.

Theorem 2. *Every solution of problems (1)–(2), (3)–(4), and (5)–(6) is Lyapunov unstable.*

For large values of n the following is valid.

Theorem 3. *Problems (2)–(1), (3)–(4), and (5)–(6) have no nontrivial solutions if* $n \geq 5$. *Every solution* $y = y(x)$, $z = z(x)$ *of equations (1) and (5), under the conditions* $y(0) > 0$, $y'(0) = 0$ *and* $z(0) > 0$, $z'(0) = 0$, *oscillates about the straight lines* $y = 1$ *and* $z = 1$, *remaining positive. Every solution* $\eta = \eta(x)$ *of equation (3), under the condition* $\eta(0) = 0$ *and* $\eta'(0) > 0$, *oscillates about the straight line* $\eta = x$, *remaining positive.*

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Note: Figure translations are in progress. See original paper for figures.