ON BOUNDARY-VALUE PROBLEMS FOR DIFFERENTIAL AND INTEGRO-DIFFERENTIAL EQUATIONS IN THE SPACES $W_{p_1\ldots p_k}^{m_1\ldots m_k}$, $W_{p_1\ldots p_k,b_1\ldots b_n}^{m_1\ldots m_k}$

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Abstract

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ON BOUNDARY-VALUE PROBLEMS FOR DIFFERENTIAL AND INTEGRO-DIFFERENTIAL EQUATIONS IN THE SPACES $W^{m_1 \ldots m_k}_{p_1 \ldots p_k}$, $W^{m_1 \ldots m_k}_{p_1 \ldots p_k, b_1 \ldots b_n}$

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The idea of the investigation is as follows: to a functional space there is put in correspondence a class of partial differential equations determined by its metric. The properties of the functions of this space lead to natural formulations of boundary-value problems for the indicated equations and to the solution of the latter (the variational method, etc.). If the spaces are "weighted," then the equations generated by their metric are degenerate. In the space $x_1 \ldots x_n$ let there be given a domain D with boundary

$$S = \sum_{s=1}^{n} S_{n-s}$$

–pieces of manifolds of dimension $n - s$. In D the functions

$$W^{m_1 \ldots m_k}_{p_1 \ldots p_k} = \prod W^{m_i}_{p_i} \left(p_i > (np_{i+j}) : [\, n - (m_{i+j} - m_i) p_{i+j} \,] \right)^* .$$

Proposition 1. If D is bounded, a function from $W^{m_1 \ldots m_k}_{p_1 \ldots p_k}$ can be characterized by assuming that it is summable in D, and that all its generalized derivatives (in the sense of S. L. Sobolev) of orders m_i are summable in D, respectively to the powers p_i $(i = 1, 2, \ldots, k)$. The space is complete.

Proposition 2. On $S_{n-s} \Subset D + S$ a function u from $W^{m_1 \ldots m_k}_{p_1 \ldots p_k}$ and all its derivatives up to order

$$\max(m_i - [s/p_i] - 1) = m_\gamma - [s/p_\gamma] - 1$$

inclusive have defined (up to a set of measure zero) traces (limiting values)

$$\varphi^{l}_{\alpha_1 \ldots \alpha_n} \big|_{S_{n-s}} \in \mathcal{L}_{q_\gamma}(S_{n-s}),$$

where 1) $q_\gamma \le p_\gamma(n-s)/[\,n-p_\gamma(m_j-l)\,]$ if $n > p_\gamma(m_\gamma - l)$; 2) $q_\gamma > 1$ is any number if $n = p_\gamma(m_\gamma - l)$; 3) one may take C if $n < p_\gamma(m_\gamma - l)$.

Example. $W^{2,1}_{2,1/2}(D_5)$, $\partial u/\partial x_i\big|_{S_4} \in \mathcal{L}_{8/3}, \dots,\ u\big|_{S_1} \in \mathcal{L}_9$. Hence the difference between the embedding theorems for $W^{2,1}_{2,1/2}(D_5)$ and $W^2_2, W^1_{1/2}(D_5)$ is visible.

Proposition 2a. The embedding operator in Proposition 2 is completely continuous in $\mathcal{L}_{q^*_\gamma}(S_{n-s})$, $q^*_\gamma < q_\gamma$.

The spaces $W^{m_1\dots m_k}_{p_1\cdots p_k}$ are defined in an unbounded D. The functions u and their derivatives up to order m_i inclusive are summable in D to the powers p_i.

Proposition 2b. Proposition 2 is valid in an unbounded D. Below W are also defined in unbounded domains.

Proposition 3. Let: 1) $S_{n-s} \Subset D + S$; 2) $\varphi^l_{\alpha_1\dots\alpha_n}\big|_{S_{n-s}}$ be the trace on S_{n-s} of the derivative $u^l_{\alpha_1\alpha_n}$ ($l \le m_\gamma - [s/p_\gamma] - 1$) of the function u ("admissible" for $\varphi^l_{\alpha_1\dots\alpha_n}$) from $W^{m_1\dots m_k}_{p_1\cdots p_k}$. The set of those admissible for $\varphi^l_{\alpha_1\dots\alpha_n}$ is an infinite set M. The set M is closed in $W^{m_1\dots m_k}_{p_1\cdots p_k}$.

Let: 1)

$$S = S^3 + S^c = (S^3_{n-1} + \dots + S^3_0) + (S^c_{n-1} + \dots + S^c_0)$$

(S^c is the "free" part of S). 2)

$$W^{m_1\dots m_k}_{p_1\cdots p_k}(S^3, 0)$$

are functions W equal to zero on S^3 together with their derivatives, ending with order $m_\gamma - [s/p_\gamma] - 1$ (by 2, 2b). 3)

$$W^{m_1\dots m_k}_{p_1\cdots p_k}(S^3,0) = W^{m_1\dots m_k}_{p_1\cdots p_k}(0), \quad S^c = 0; \quad W^{m_1\dots m_k}_{p_1\cdots p_k}(0) \Subset W^{m_1\dots m_k}_{p_1\cdots p_k}(S^3,0)$$

and is closed in it.

The proof of the propositions follows from the properties of the functions W^m_p.

* Definition see ([1]), domain and boundary ([1–5]), history of the question ([1–5, 7–9]).

I shall present problem 1 in such a way that all the specific features in the formulation of boundary-value problems in W will be visible (boundary conditions, nonlinearities, etc.). Some notation from ([1,4]) is used, as well as the following.

$F_m^p(x, u \dots u^m_{\alpha_1\dots\alpha_n})\big|_{S_{n-s}} = F_m^p(u)\big|_{S_{n-s}}$ is a form admissible on S_{n-s} in W_p^m (or a homogeneous function) of degree p in the indicated arguments;

$$D_m^{p-1,1}[F(u,\xi)]_{S_{n-s}} = \int_{S_{n-s}} \cdots \int \sum \frac{\partial F_m^p}{\partial u^l_{\alpha_1\dots\alpha_n}} \xi_{\alpha_1\dots\alpha_n}\, ds.$$

Problem 1. In $W^{m_1\dots m_k}_{p_1\dots p_k}$, find a function u satisfying the equation

$$\sum_{i=1}^{k} \sum_{\nu=0}^{m_i} \sum_{p_\nu, q_\nu \geqslant 1}^{q_i} \left(D_\nu^{p_\nu}[F(u)]\right)^{q_\nu - 1} D_\nu^{p_\nu - 1,1}[F(u,\xi)] = 0 = J^{p-1,1}(u,\xi);$$

$$p_{m_i} = p_i; \quad q_{m_i} = \tau_i; \quad \xi \in W^{m_1\dots m_k}_{p_1\dots p_k}(0); \quad q_i \leqslant p_i n/[\, n - p_i(m_i - \nu)\,] \qquad (1)$$

and so on (see Theorems 2 and 26).

The integro-differential equation has the form

$$\sum_{i=1}^{k} \left\{ \left(\int_D \cdots \int F^{p_i}_{m_i}(u)\, dV \right)^{\tau_i - 1} \sum_{0,\ \Sigma\alpha = l}^{l = m_i} (-1)^l \left(\frac{\partial F^{p_i}_{p_i}(u)}{\partial u_{\alpha_1\dots\alpha_n}} \right)^{l}_{\alpha_1\dots\alpha_n} \right\} + \cdots = 0. \quad (1')$$

Here u must satisfy the boundary conditions

$$1. \quad J^{p-1,1}(u,\tilde{\xi}) + \sum_{i,s=1}^{k,n} \sum_{\nu=0}^{m_i - (s/p_i) - 1} \sum_{p_{\nu,s}, q_{\nu,s} \leqslant 1}^{q_{\nu,i,s}} \left(D_\nu^{p_{\nu,s}}[F(u)]\right)^{q_{\nu,s} - 1}_{S^c_{n-s}} \times$$

$$\times D_\nu^{p_{\nu,s} - 1,1}[F(u,\tilde{\xi})]\Big|_{S^c_{n-s}} = 0; \quad \tilde{\xi} \in W^{m_1\dots m_k}_{p_1\dots p_k}(S^3, 0). \qquad (2)$$

It is possible that in (2) the integrals over S^c_{n-s} are absent (the second boundary condition). It is possible that $S^3 = 0$ (the condition

$$\int_D \cdots \int u\, dV = 0$$

and so forth ([1])).

2. On the pieces S^3_{n-s}, for the function u the boundary conditions are as follows:

$$u\big|_{S^3_{n-s}} \quad \varphi_0(S^3_{n-s}) \ \cdots \ \left.\frac{\partial^{m_\gamma-[s/p_\gamma]-1}u}{\partial x_n^{m_\gamma-[s/p_\gamma]-1}}\right|_{S^3_{n-s}} = \varphi_{0,0,\dots,m_\gamma-[s/p_\gamma]-1}(S^3_{n-s}). \quad (3)$$

Here $\varphi_{\alpha_1-\alpha_n}(S^3_{n-s}) \subset \mathcal{L}_{q_\gamma}(S^3_{n-s})$ (see Propositions 2, 26) is a necessary condition for solving the problem by the variational method*. (2) is the first variation of the functional

$$Z(u) = \sum_{i=1}^{k}\left\{\left(D^{p_i}_{m_i}[F(u)]\right)^{\tau_i} + \sum_{s=0}^{n}\sum_{\nu=0}^{m_i-[s/p_i]-1}\sum_{p_{\nu,s},q_{\nu,s}\geqslant 1}^{q_{\nu,i,s}}\left(D^{p_{\nu,s}}_{\nu}[F(u)]\right)^{q_{\nu,s}}_{S^c_{n-s}}\right\}. \quad (2')$$

A function $u \in W^{m_1\dots m_k}_{p_1\dots p_k}$ for which the functional $(2')$ has meaning and which satisfies conditions (3), according to assumptions 2, 26, is an admissible function of problem 1. Conditions are imposed on $(2')$ ensuring its lower bound in the class of admissible functions (see ([1,4])), etc.**

Proposition A. Among the admissible functions of problem 1 there exists a unique one that gives the minimum of the functional $(2')$. It is the solution of problem 1.

* On sufficient conditions for a number of cases of first boundary-value problems, see ([1,6–8]).

** $F^p_m = \sum_{\Sigma\alpha=m} A_\alpha(x)\,|u^m_{\alpha_1\dots\alpha_n}|^p + \widetilde{F}^p_m(u) = \Phi^p_m + \widetilde{F}^p_m; \quad A_\alpha \geqslant a; \quad F^p_m \geqslant b\Phi^p_m; \quad a,b>0.$

Example. D is a domain in the plane x_1, x_2; $S = S^3_1 + S^3_2 + S^c$ is the boundary of D. In $W'_{2,b}$, find a function u such that (r^b_2 is from D to D^3_2):

$$\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\left(r^b_2\frac{\partial u}{\partial x_i}\right) + c(x)\left(\iint_D |u|^p\,dV\right)^{q-1} u|u|^{p-2}\,dV = 0 \quad \text{in } D.$$

$$1)\quad r^b_2\left.\frac{\partial u}{\partial n}\right|_{S^c} + \beta(S)\left(\int_{S^c}|u|^{p_1}\,dS\right)^{q-1} u|u|^{p_1-1} = 0; \quad 2)\ u\big|_{S^3_1} = \varphi(S); \quad 3)\ u\big|_{S^3_2} = 0.$$

Problem 2. Among functions $u \in W^{m_1\dots m_k}_{p_1\dots p_k}$ satisfying the boundary conditions (3), according to Propositions 2, 2b (admissible functions), find one that gives the minimum to the functional

$$Z(u) = J\left(\underbrace{\int_D\cdots\int}_{n} F^{p_1}_{m_1}(u)\,dV \ \cdots \ \underbrace{\int_{S^c_{n-s}}\cdots\int}_{n-s} F^{p_s}_{s}(u)\,dS_{n-s}\cdots u(Q_{0,i})\right). \quad (4)$$

Here the functional (4) satisfies the conditions:

$$1.\quad J\left(\int f\left(\frac{\varphi+\psi}{2}\right)dt\right)+J\left(\int f\left(\frac{\varphi-\psi}{2}\right)dt\right)\le J\left(\int F\left({}^1/_2(\gamma(\varphi)+\gamma(\psi))\right)dt\right), \tag{5_1}$$

$f(\ ) = F(\gamma(\ ))$. This inequality is valid for broad types of functions J, F and γ. For example: 1) $J = (\ )^q$, $F = (\ )^\tau$, $\gamma = |\ |^p$; 2) $J = (\ )^q$, $F = (\ )$, $\gamma = \gamma(\ )$, $J = J(\ )$, etc.*

$$2J\left(\underbrace{\int_D\cdots\int}_{n} F_{m_1}^{p_1}(u)\,dV\cdots\right)\ge A\left\{\sum_{i=1}^{k}\underbrace{\int_D\cdots\int}_{n}\sum_{\Sigma\alpha=m_i}\left|u_{\alpha_1\dots\alpha_n}^{m_i}\right|^{p_i}dV\right\}. \tag{5_2}$$

3. Some smoothness, lower semicontinuity.

The solution is carried out by the variational method. The passage from Problem 2 with the functional $Z(u)$ to the corresponding boundary-value problem with a generalized solution is performed taking into account inequalities generalizing those given below. Instead of (4), one may consider in general a functional $Z(u)$ defined on $W_{p_1\dots p_k}^{m_1\dots m_k}$, with conditions analogous to (5). When the functional (4), (2′) corresponds to equation (1), Proposition A is proved with the aid of Propositions 2, 2b, 3, relations (5_1), and a number of inequalities. I give some of them.

$$1.\quad\left[\sum_{i=1}^{k}D^{p_i}(\varphi_i)\right]^q-\left[\sum_{i=1}^{k}D^{p_i}(\varphi_i-\psi_i)\right]^q\le q\left[\sum_{i=1}^{k}D^{p_i}(\varphi_i)\right]^{q-1}\sum_{i=1}^{k}p_{iD}^{p_i-1,1}(\varphi_i,\psi_i)$$

$$+\sum c\left[\sum D^{p_i}(\varphi_i)\right]^{q_1}\left[\sum D^{p_i-1,1}\right]^{q_2}\left[\sum D^{p_i-2,2}\right]^{q_3}\left[\sum D^{p_i}(\psi_i)\right]^{q_4},$$

$\sum q_i \ge 2$, q an integer, and $\sum q_i \ge q-([q]-1)$ in the other cases; $q_i = 0, 1, \dots, q$.

$$2.\quad\int|\varphi-\psi|^p\,dt\le c\left\{\int\varphi|\varphi|^{p-q-1}\left(\varphi|\varphi|^{q-1}-\psi|\psi|^{q-1}\right)dt+\right.$$

$$\left.+\int\psi|\psi|^{p-q-1}\left(\psi|\psi|^{q-1}-\varphi|\varphi|^{q-1}\right)dt\right\};\quad p\ge 2;\quad 1\le q\le p-1.$$

On the smoothness of the solution of equation (1). Consider $W_{p_1\dots p_k=2}^{m_1\dots m}$ ($\max m_i = m_k = m$). We write equation (1) in the form ($\tau_k = 1$)

$$\underbrace{\int_D\cdots\int}_{n} u\,\mathcal{L}^m(\xi)\,dV+\sum_{\nu=0}^{m-\lambda}\sum_{p_\nu,q_\nu}\left(D_\lambda^{p_\nu}[F(u)]\right)^{q_\nu-1}D_\nu^{p_\nu-1,1}[F(u,\xi)]=0, \tag{1_1}$$

* Generalization of Clarkson' s inequality.

$0 \le n \le m,\ \mathcal{L}^m(\xi) = \sum_{\Sigma\alpha=m} (A_\alpha(x)\xi^m_{\alpha_1\cdots\alpha_n})^m_{\alpha_1\cdots\alpha_n}$, A_α are m times differentiable. It is possible that $A_\alpha = \widetilde{A}_\alpha B$; in degenerating equations $B(x) = \prod r^{b_s}_{n-s}$, where $r^{b_s}_{n-s}$ is the distance to S_{n-s} to the power b_s. Let the operator $\mathcal{L}^m(\xi)$ be self-adjoint elliptic and admit a fundamental solution.

Proposition 1a. *Let us assume that the coefficients in the forms F^p_ν are bounded in any $\overline{D}_1 \Subset D$. Then the function u (the solution of (1_1)) will be continuous (more precisely, equivalent to the latter) in D, together with its derivatives up to order $m+n-1$ inclusive.*

Proposition 1b. *Suppose that the coefficients $A_{l,q}$ in the terms*

$$A_{l,q} \prod |u^{l_i}_{\alpha_1\cdots\alpha_n}|^{q_i} \prod (u^{l_j}_{\alpha_1\cdots\alpha_n})^{q_j}$$

of the forms F^p_ν have piecewise-continuous derivatives in D up to order

$$\max(l_1, \dots, l_r, \dots, l_H) + 1$$

*(in the case of the function F^p_ν the corresponding smoothness and the T-properties are required—substitution of a function into a function [5]). Then the function u satisfies the equation **

$$\mathcal{L}^m_{b_1\cdots b_n}(u) + F\left(Q, u, \dots, u^{2m-\lambda}_{\alpha_1\cdots\alpha_n}, \dots \overbrace{\int_D \cdots \int}^{n} F^p_\nu(u)\, dV\right) = 0. \qquad (1_2)$$

Propositions 4a and 4b are proved by a generalization (development) of the method of S. L. Sobolev [2,5] ($\Delta^m(u) = 0$). This generalization was given by the author partially also in [1,4]. For clarity of exposition let us set $\mathcal{L}^m(\ ) = \Delta^m(\ )$. From (1_1) we obtain the equations

$$u^l_{\alpha_1\cdots\alpha_n} = \overbrace{\int_D \cdots \int}^{n} [\,]\, u^k_{\alpha_1\cdots\alpha_n}\, dV + \overbrace{\int_D \cdots \int}^{n} \sum \frac{\partial F}{\partial u^l_{\beta_1\cdots\beta_n}} \left(\omega\left(\frac{r}{h}\right)\right)^{k+l}_{\alpha_1+\beta_1,\dots,\alpha_n+\beta_n} dV. \qquad (1_3)$$

Here [ ] is the kernel of S. L. Sobolev. The second integral is the "potential" term. By investigating (1_3) the smoothness properties of the solution of equation (1_1) are established. The case when in (1_1) all

$$A_\alpha(x) = B(x)\frac{m}{\alpha_1! \cdots \alpha_n!}$$

(in degenerating equations $B(x) \sim\sim \prod_{s=1}^{n} r_{n-s}^{b_s}$) is reduced to the same question. In the functional $(4), (2')$ the new function is $V = \sqrt{A}\,u$. In the general form of the operator $\mathcal{L}^m(u)$, its fundamental solution is used. There are also other cases of differentiability of solutions of problem 1. The investigation of differential properties of generalized solutions of a number of cases of the main problem 1 can also be carried out by other methods (finite differences, ellipticity of equations, smoothness of $F_m^p(u)$, etc.).** The problems in $W_{p_1 \dots p_k, b_1 \dots b_n}^{m_1 \dots m_k}$ are formulated and considered analogously to what was set out above. Problem 2 and the corresponding boundary problem naturally extend to other functional spaces (with fractional indices of S. L. Sobolev, K, and others). For detailed literature see ([1],[5],[7]).

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* Clarifications: the theorems are formulated also with the Hölder condition; for $\min(q_1, \dots, q_n) \le \max(l_1, \dots, l_n) + 1$ (q_i not an integer) Proposition 1b will look somewhat different.

** This method was also known earlier (works other than the present one)– Aronszajn, A. Douglis, S. M. Nikol' skii, L. Nirenberg, and others.

Note: Figure translations are in progress. See original paper for figures.