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### MATHEMATICS


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# LOCALLY BICOMPACT RINGS WITHOUT ZERO DIVISORS

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Locally bicompact division rings have been well studied in the works of Kaplansky, Kowalsky, and Pontryagin. A connected locally bicompact division ring is either the field of real numbers, or the field of complex numbers, or the division ring of quaternions. If, however, the division ring is totally disconnected, then it is a finite-dimensional vector space either over the field of $p$-adic numbers or over a field of power series. In the present paper we investigate the structure of locally bicompact rings with identity and without zero divisors. In what follows, by the word ring we shall always mean a ring with identity and without zero divisors. Since, as Kaplansky showed, connected locally bicompact rings are division rings, we shall consider only totally disconnected ones. It turns out that a significant part of this class of rings consists of bicompact local rings.

**Theorem 1.** *Every locally bicompact normed ring without zero divisors is either a division ring or a bicompact local ring.*

It is not difficult to show that, depending on the characteristic, a locally bicompact normed ring contains either the ring of $p$-adic integers, or the ring of power series in one variable over the field with $p$ elements. However, not every bicompact ring must be finite-dimensional over them. An example is the usual ring of power series in one variable over the ring $Z^p$ of $p$-adic integers, normed as follows: 1) $|at^i| = |a||t|^i$, where $a \in Z^p$ and $|t| = \alpha < 1$; 2) $\left|\sum_i a_i t^i\right| = \max_i |a_i t^i|$, $a_i \in Z^p$. Rings in which the finite-dimensionality condition is fulfilled are precisely the closed subrings of locally bicompact division rings.

**Theorem 2.** *Let $K$ be a locally bicompact ring without zero divisors. Then the ring $K$ can be embedded in a locally bicompact division ring if and only if the ring $K$ is normed and finite-dimensional over the ring $Z^p$ of $p$-adic integers (characteristic zero) or over the ring of power series in one variable over the field $Z_p$ with $p$ elements (characteristic $p$).*

Let us proceed to the study of bicompact rings. It is easy to show that every bicompact ring is local. Complete local rings, beginning with Cohen, have been studied by many authors. Their efforts were directed toward decomposing a ring, under one or another set of conditions, into a sum of a "coefficient field" and the radical (the equicharacteristic case). These investigations were continued by Tedd, who proved, in a certain sense, an analogous decomposition theorem for the unequal-characteristic case. In all these works commutative rings were considered.

**Theorem 3.** *Let $K$ be a bicompact ring without zero divisors. If the characteristic of $K$ is $p$, then in the ring $K$ there exists a field $P$, isomorphic to the field $K/R$ ($R$ is the radical of the ring $K$), and all such fields are conjugate to one another by means of invertible elements of the ring $K$.*

If the characteristic of $K$ is equal to 0, then in the ring $K$ there is a commutative closed subring $S$ with 1, satisfying the following conditions:
1) $S + R/R = K/R$; 2) $S$ does not contain closed subrings with the property 1. All subrings $S$ with these properties are conjugate by means of invertible elements of the ring $K$.

In what follows we shall call the ring $S$ the **coefficient ring**, and the field $\overline{P}$ the **coefficient field**.

The possibility of splitting in the case of characteristic $p$ is contained in Zelinsky' s work, but even in this case our theorem is more precise, since it asserts not only existence, but also conjugacy of the coefficient fields. The method of proof is different from Zelinsky' s, and is uniform for both cases. The proof uses another well-known theorem of Zelinsky stating that a bicompact ring is the inverse limit of the spectrum of factor rings $K/I_\alpha$, where $\{I_\alpha\}$ is the set of all open ideals of the ring $K$.

Let $P^*$ be the multiplicative group of the field $P = K/R$, and let $K^*$ and $K^*_\alpha$ be the multiplicative groups of invertible elements of the ring $K$ and of the finite ring $K_\alpha = K/I_\alpha$, respectively. Then

$$K^*_\alpha = A_\alpha \cdot N_\alpha,$$

where $A_\alpha \simeq P^*$, and $N_\alpha$ is a normal divisor, and all groups $A_\alpha$, isomorphic to the group $P^*$, are conjugate in the group $K^*_\alpha$. These groups $A_\alpha$ can be chosen so that they form an inverse spectrum; then the group

$$A = \lim_{\leftarrow} A_\alpha$$

is isomorphic to the group $P^*$ and is contained in the ring $K$. If $A'$ is another group isomorphic to the group $A$, then it can be shown that $A$ and $A'$ are conjugate in the group $K^*$. Taking the subring generated by the group $A$ and either the field $Z_p$ of $p$ elements, or the ring $Z^p$ of integral $p$-adic numbers, it is not hard to show that one obtains either the coefficient field $\overline{P}$, or the coefficient ring $S$, respectively, indicated in Theorem 3. Their conjugacy follows immediately from the conjugacy of the groups isomorphic to the group $P^*$ in the group $K^*$.

Thus, a bicompact ring contains a coefficient ring (or field). Over this coefficient ring (field) one constructs in a special way a ring of power series in a countable or finite number of noncommuting variables. We shall not describe the construction process itself; we note only that the coefficient ring (field) need not lie in the center of the ring of power series. The constructed ring of power series can be bicompactly normed. In the class of bicompact rings satisfying the first axiom of countability, it will be free.

**Theorem 4.** *Let $K$ be a bicompact ring without zero divisors, satisfying the first axiom of countability, and let $A$ be a coefficient ring (field) contained in $K$. Then the ring $K$ is a continuous homomorphic image (and even a factor ring) of the ring $\mathfrak{S}(A)$ of power series in a countable number of variables over the ring $A$.*

It can be shown that if, moreover, in the ring $K$ there exist $m$ topological generators ($m$ finite), then $K$ is a factor ring of the ring $\mathfrak{S}_m(A)$ of power series in $m$ variables over the ring $A$. From this it is no longer difficult to obtain the following theorem.

**Theorem 5.** *Every algebraic isomorphism of bicompact rings without zero divisors with a finite number of topological generators and satisfying the first axiom of countability is topological.*

In conclusion, let us dwell on the following question. As was indicated above, every locally bicompact skew field is normable. The question remained open whether every normable locally bicompact ring without zero divisors is normable. It turned out that it is not; namely, an example can be constructed of a bicompact ring without zero divisors which does not satisfy the first axiom of countability, and therefore is not normable and is not even pseudonormable.

**Example.** Let $M_1$ be an uncountable set, and let $R_\alpha$ ($\alpha \in M_1$) be a collection of rings isomorphic to one and the same topological ring $R$

with 1 and without zero divisors, and let $M_k$ be the set of $k$-tuples $(a_1, a_2, \dots, a_k)$, where $a_1, a_2, \dots, a_k \in M_1$. Denote by

$$L_k = \sum_{(a_1,\dots,a_k)\in M_k} R(a_1, \dots, a_k),$$

where $k$ is a natural number, the complete direct sum of the additive groups of rings $R(a_1, \dots, a_k) \cong R$, endowed with the Tikhonov topology, and by

$$L(R) = \sum_{k=1}^{\infty} L_k$$

the complete direct sum of the groups $L_k$, again with the topology of a direct sum. Thus, $L(R)$ is an abelian topological group that is the direct sum of a countable set of groups isomorphic to the additive topological group of the ring $R$; $L_k$ is called the homogeneous component of degree $k$. Define multiplication in the group $L(R)$ so as to turn it into a topological ring. Let $u \in L_m$ and $v \in L_n$. Then in the element $uv$, at the place determined by the sequence $(a_1, \dots, a_m, a_{m+1}, \dots, a_{m+n})$, there must stand the element $z = xy$, where $x = x(a_1, \dots, a_m)$ is equal to the element standing at the place determined by the sequence $(a_1, \dots, a_m)$ in the element $u$, and $y = y(a_{m+1}, \dots, a_{m+n})$ is determined analogously. If $u$ and $v$ are nonhomogeneous, with

$$u = \sum_{i=1}^{\infty} u_i, \qquad v = \sum_{j=1}^{\infty} v_j,$$

then put

$$uv = \sum_{k=1}^{\infty} \left( \sum_{i+j=k} u_i v_j \right).$$

It can be shown that $L(R)$ is a topological ring without zero divisors. The ring $L(R)$ is easily made into an $R$-module by defining multiplication by elements of $R$ componentwise. Now put $K(R) = L(R) + R$, where $K(R)$ is the direct sum of the additive groups of the rings $L(R)$ and $R$ with the topology of a direct sum. Define multiplication in the group $K(R)$, namely set

$$(l_1 + r_1)(l_2 + r_2) = (l_1 l_2 + l_1 r_2 + r_1 l_2) + r_1 r_2,$$

where $l_1, l_2 \in L(R)$ and $r_1, r_2 \in R$. Since $L(R)$ is an $R$-module, $(l_1 l_2 + l_1 r_1 + r_1 l_2) \in L(R)$, and, consequently, the multiplication is defined correctly. If $R$ is a bicompact ring, then one can verify that $K(R)$ will be a bicompact ring with 1 and without zero divisors. Since every neighborhood of zero in the ring $K(R)$ is determined by a finite number of indices, the intersection of a countable number of neighborhoods of zero is not equal to zero, i.e., $K(R)$ cannot be normable.

Thus, not every bicompact ring is pseudonormable; however, the following theorem is true.

**Theorem 6.** *A bicompact ring without zero divisors is pseudonormable if and only if the first axiom of countability holds in it.*

The necessity of the condition is obvious, while the sufficiency follows from Theorem 4.

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## REFERENCES

1. A. Geddes, Proc. London Math. Soc. (3), **6**, 343 (1956).

2. D. Zelinsky, Duke Math. J., **21**, No. 2, 315 (1954).

3. J. S. Cohen, Trans. Am. Math. Soc. (59), No. 1, 54 (1946).

4. L. S. Pontryagin, *Continuous Groups*, Moscow, 1954.

*Note: Figure translations are in progress. See original paper for figures.*