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Abstract

Full Text

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ON CONDITIONS FOR NONLINEAR STABILITY OF PLANE STATIONARY CURVILINEAR FLOWS OF AN IDEAL FLUID

(Presented by Academician I. G. Petrovskii, 20 XI 1964)

§ 1. In the present note we indicate some hydrodynamic consequences of three simple theorems from the theory of ordinary differential equations. In particular, sufficient conditions are obtained for the stability of plane stationary curvilinear flows of an incompressible inviscid fluid with respect to finite perturbations.

For infinitesimal perturbations of plane-parallel flows, the stability condition was already indicated by Rayleigh: it is sufficient that the velocity profile have no points of inflection ([1–3]). It turns out that Rayleigh' s condition is also sufficient for stability with respect to finite perturbations; moreover, our results imply the stability of certain flows with one point of inflection.

By a perturbation we mean a small change of the initial velocity field compatible with the condition of incompressibility. All functions considered, including the perturbations, are assumed to be differentiable the required number of times. For simplicity, we consider perturbations that do not change the value of the circulation of the velocity along each boundary curve.*

By stability we mean "Lyapunov stability." In other words, we prove that if at the initial instant the perturbation $\delta\psi$ is small, then the velocity field of the perturbed motion remains close to the unperturbed one at all times ("closeness" is understood in the sense of the metric (11)).

§ 2. **Conditional extrema and positions of equilibrium.** The source of Theorems 1 and 2 is the well-known argument proving the stability of Eulerian rotation of a rigid body about the major and minor axes of inertia (see ([4])). Let the system of differential equations

$$\dot{x} = f(x) \qquad (x = x_1, \dots, x_n) \tag{1}$$

have single-valued first integrals $E(x)$; $F_1(x), \dots, F_k(x)$ $(1 \leq k < n)$. Consider the level set

$$F_i(x) = c_i \qquad (i = 1, \dots, k). \tag{2}$$

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Let x_0 be a point of conditional extremum of the function E under the conditions (2), i.e., at the point x_0, for suitable Lagrange multipliers λ,

$$dH = dE + \lambda_1 dF_1 + \dots + \lambda_k dF_k \equiv 0. \quad (3)$$

Theorem 1. *The point x_0 is an equilibrium position of the system (1), if one of the following two conditions is fulfilled:*

1. *x_0 is a point of conditional maximum or minimum (at least local).*
2. *The extremum is nondegenerate, i.e., the quadratic form in dx*

$$d^2H = d^2E + \lambda_1 d^2F_1 + \dots + \lambda_k d^2F_k \quad (4)$$

is nondegenerate on the subspace $dF_1 = 0, \dots, dF_k = 0$.

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* In the case of plane-parallel flows, from stability with respect to such perturbations one can easily derive stability with respect to all perturbations. Indeed, a perturbation changing the circulations may be regarded as a circulation-preserving perturbation of a nearby stationary flow. Similarly, the restriction on the flux in Example 2 of § 6 is inessential.

According to Theorem 1, the following **variational principle** holds: a stationary flow has extremal energy in comparison with flows having equimeasurable vorticity.

Theorem 2. *If both conditions of Theorem 1 are satisfied, then the equilibrium position $x = x_0$ is stable.*

The following simple theorem is a generalization of the Poincaré–Lyapunov theorem on the characteristic numbers of Hamiltonian systems ([5,6]).

Theorem 3. *If the linear real system $\dot{x} = Ax$ has as a first integral the nondegenerate quadratic form (Bx, x), then the set of eigenvalues λ of the operator A is symmetric (counting multiplicity) with respect to the real and imaginary axes. Moreover, the number of points λ lying strictly in the left half-plane,* $\operatorname{Re}\lambda < 0$, *does not exceed the smaller of the inertia indices ν_+, ν_- of the quadratic form (Bx, x).*

§ 3. Equations of motion and their first integrals.

Let $\psi(x, y; t)$ be the stream function, $v_1 = \partial\psi/\partial y$, $v_2 = -\partial\psi/\partial x$ the components of the velocity v, and $\operatorname{rot} v = -\Delta\psi$ the vorticity.

The domain D, filled with an ideal fluid, will be regarded as a ring bounded by two smooth fixed curves Γ_1, Γ_2 (Fig. 3). The velocity at the boundary is assumed to be parallel to the boundary. Thus, at each instant t, on each

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boundary curve $\Gamma_{1,2}$ the stream function ψ takes a constant value $\psi_1 = 0,\ \psi_2 = \varphi(t)$.

Taking the curl of both sides of the equation $\dot{v} = -\operatorname{grad} p$, we find

$$\partial\Delta\psi/\partial t = -[\nabla\Delta\psi, \nabla\psi], \qquad \text{where } [\xi, \eta] = \xi_1\eta_2 - \xi_2\eta_1. \tag{5}$$

In the space of functions $\psi(x, y)$, equation (5) defines the dynamical system (1)*.

Equation (5) expresses conservation of the vorticity of a particle, $d\Delta\psi/dt = 0$. Because the fluid is incompressible, it follows from this that the vorticity is "equimeasurable" : the measure of the set of points of the domain D where $\Delta\psi < a$ does not change with time. Therefore, for any function $\Phi(\xi)$, the functional on the space $\psi(x, y)$

$$F = \iint \Phi(\Delta\psi)\, dx\, dy \tag{6}$$

is a first integral of system (5). Another integral is given by the law of conservation of the energy E, where

$$2E = \iint (\nabla\psi)^2\, dx\, dy. \tag{7}$$

The equilibrium positions of system (5) are stationary flows $\psi(x, y)$. For such flows, according to (5), $\nabla\Delta\psi$ and $\nabla\psi$ are collinear, i.e.

$$\psi = \Psi(\Delta\psi), \tag{8}$$

provided $\nabla\Delta\psi \neq 0$, i.e. provided the vorticity has no extremal values inside the domain D**.

§ 4. Calculation of the first and second variations.

Consider the class of functions $\psi(x, y)$ that are constant on each of the boundary curves Γ_1, Γ_2 and satisfy the conditions

$$\oint_{\Gamma_1} \frac{\partial\psi}{\partial n}\, ds = C_1, \qquad \oint_{\Gamma_2} \frac{\partial\psi}{\partial n}\, ds = C_2. \tag{9}$$

The corresponding flows have fixed values of the circulation of the velocity $C_{1,2}$ on each component of the boundary. We shall seek conditional

* Since the domain D is doubly connected, the vorticity equation (5) does not uniquely determine the solution $\psi(x, y; t)$ from the initial condition $\psi(x, y)$ and

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Fig. 1 and Fig. 2

Figure 1: Fig. 1 and Fig. 2

the boundary values $\psi|_{\Gamma_1} = 0,\ \psi|_{\Gamma_2} = \varphi(t)$. To equation (5) one must also add the conservation law for the circulation $\oint_{\Gamma_1} v\,ds$, which follows from the single-valuedness of the pressure p. The value of the fluid discharge $\varphi(t)$ does not enter into the boundary conditions, but is determined by the solution.

** For parallel flows this condition means that the velocity profile has no inflection points.

an extremum of E with fixed F, by Lagrange' s formula (3)

$$\delta H = \delta \iint \left\{ \frac{1}{2}(\nabla\psi)^2 + \lambda\Phi(\Delta\psi) \right\} dx\,dy = 0. \tag{10}$$

Integrating by parts, we obtain, for $u = \psi - \lambda\Phi'(\Delta\psi)$,

$$\delta H = -\iint_D \Delta u\,\delta\psi\,dx\,dy + \oint_\Gamma \frac{\partial u}{\partial n}\,\delta\psi\,ds + \oint_\Gamma \lambda\Phi' \frac{\partial\delta\psi}{\partial n}\,ds.$$

For $u = 0$, and on the basis of (9), the right-hand side vanishes. Thus is proved

Fig. 1 Fig. 2

Theorem I. *The stationary flow (8) is a point of conditional extremum of E with fixed F and a point of absolute extremum of the functional $H = E + \lambda F$, defined by formulas (6), (7), where*

$$\lambda\Phi(\xi) = \int^{\xi} \Psi(\eta)\,d\eta.$$

Computing the quadratic form $\delta^2 H$, we arrive at the basic formula

$$2\delta^2 H = \iint \left\{ (\nabla\delta\psi)^2 + \frac{\nabla\psi}{\nabla\Delta\psi}(\Delta\delta\psi)^2 \right\} dx\,dy. \tag{11}$$

§ 5. **Stability conditions.**

Theorem II. *A stationary flow with stream function $\psi(x, y)$ is stable if the form (11) is sign-definite for $\delta\psi|_{\Gamma_1} = 0,\ \delta\psi|_{\Gamma_2} = C_0$.* Indeed, the first integral $H = E + \lambda F$ has ψ as a point of local maximum or minimum (in the class of smooth $\psi(x, y)$ with fixed boundary values (9)). This also proves the stability of ψ with respect to smooth perturbations preserving the circulation of velocity on the boundary.

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Fig. 3

Figure 2: Fig. 3

If, without being concerned with justification, one applies theorem 3 to the system (5) linearized in a neighborhood of the stationary flow (8), then we obtain

Theorem III. *The set of eigenvalues λ of the problem on the stability of a plane flow of an ideal fluid is symmetric with respect to both axes: the real and the imaginary.*

Remark. This conclusion, in the case of a plane-parallel flow, is justified by a direct analysis of the Orr–Sommerfeld equation for $\nu = 0$ (the reality of the equation corresponds to symmetry with respect to the imaginary λ-axis; preservation of the equation under replacement of α by $-\alpha$ corresponds to symmetry with respect to $\lambda = 0$). It seems to me that the spectrum is symmetric under considerably more general assumptions than (8), for example, for three-dimensional flows of an inviscid fluid.

§ 6. **Examples.** Let us apply the results of § 5 first of all to flows parallel to the x-axis in the strip* $\{Y_1 \leq y \leq Y_2,\ x \bmod X\}$. We have

$$\psi = \psi(y), \qquad \nabla\psi = v, \qquad \Delta\psi = v', \qquad \nabla\Delta\psi = v''. \tag{12}$$

* We consider the flows to be X-periodic in x and identify the points $(x + X, y)$ and (x, y). The results do not depend on the magnitude of X.

Example 1. Flows without inflection points ($v'' \neq 0$).

Choose an inertial coordinate system in which the sign of v everywhere coincides with the sign of v''. Then the form (11) is positive definite. Thus, *all flows without inflection points are stable.*

Fig. 3

Example 2. Flows with one inflection point ($v''(0) = 0$).

Suppose that the velocity profile is symmetric with respect to the inflection point. Choose an inertial coordinate system in which the velocity of the inflection point is zero, so that $v(-y) = -v(y)$. The form (11) is positive definite if v and v'' have the same sign. Thus, *flows with a velocity profile of the type shown in Fig. 1 are stable.* For example, the flow with $v = a + by + cy^3$ is stable for $bc > 0$. As Tollmien showed ([7]), for $b = 0,\ Y_1 + Y_2 = 0$ this flow is unstable.

If v and v'' have different signs, then for stability it is sufficient that the form (11) be negative definite. Let us consider perturbations that preserve, in addition to circulation, also the total discharge of fluid of the unperturbed motion*: $\delta\psi|_\Gamma = 0$. Under this boundary condition, as is known,

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$$\iint (\nabla\delta\psi)^2\, dx\, dy \leq \frac{(Y_2 - Y_1)^2}{\pi^2} \iint (\Delta\delta\psi)^2\, dx\, dy. \tag{13}$$

From inequality (13) follows the stability of flows with

$$\left|\frac{v - v(0)}{v''}\right| > \frac{(Y_2 - Y_1)^2}{\pi^2}.$$

For example, *the flow with velocity* $v = a + b \sin y$ *is stable for* $Y_2 - Y_1 < \pi$ (Fig. 2). As Tollmien showed ([7]), for $Y_2 - Y_1 > \pi$, $Y_2 + Y_1 = \pi$ this flow is unstable.

Example 3. Flow in a curvilinear annulus of general form.

From Theorem II follows the *stability of flows with a concave velocity profile* ($\nabla\psi/\nabla\Delta\psi > 0$, Fig. 3).

If, however, the velocity profile is convex ($\nabla\psi/\nabla\Delta\psi < 0$), then from Theorem 3 and inequalities of type (13) follows the *finiteness of the set of unstable eigenvalues* ($\operatorname{Re}\lambda > 0$) of the corresponding linear problem.

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* From the law of conservation of the amount of motion it is evident that, under such a perturbation of the initial conditions, the discharge will remain equal to the discharge of the unperturbed motion for all t.

** *Note added in proof.* The author expresses gratitude to L. A. Dikii, who pointed out important articles ([8,9]), the results of which often overlap with the results of the present note.

Note: Figure translations are in progress. See original paper for figures.

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