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Abstract

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ON THE DIMENSION OF REMAINDERS IN COMPACT EXTENSIONS

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As all the spaces under consideration, so also their extensions will, by assumption, be metric with a countable base, or, equivalently, subsets of the Hilbert cube. Apparently, some results can be generalized. But we restrict ourselves to this range of spaces because at present it seems the most suitable for those questions of dimension theory from which we are trying to remove the most paradoxical features.

Theorem 1. *If the space X is compact and the space Y is not locally compact, then every compact extension $c(X \times Y)$ of the product $X \times Y$ satisfies the inequality*

$$\dim X \leq \dim(c(X \times Y) \setminus (X \times Y)).$$

Proof. Put

$$n = \dim(c(X \times Y) \setminus (X \times Y)).$$

Since Y is not locally compact, there exists a point $y_0 \in Y$ such that for every number $\varepsilon > 0$ there is an ε-neighborhood Oy_0 and an infinite sequence of distinct points $y_i \in Oy_0$ having no subsequence convergent in Y. Using the compactness of the space $c(X \times Y)$, we may at once suppose that the sequence of sets $X \times \{y_i\}$ converges to some subset K of the space $c(X \times Y)$. Then K is a compactum lying entirely in

$$c(X \times Y) \setminus (X \times Y).$$

Hence $\dim K \leq n$, and therefore there exists a finite, at most n-fold, covering of the compactum K by open (in $c(X \times Y)$) sets $G_1, \dots, G_m$, each of which has diameter less than ε. Thus, for some i_0, the set $X \times \{y_{i_0}\}$ is contained in the union

$$G_1 \cup \dots \cup G_m.$$

Consider the mapping f of the set $X \times \{y_0\}$ onto the set $X \times \{y_{i_0}\}$, putting

$$f(x, y_0) = (x, y_{i_0}).$$

Since f is an ε-displacement and X is a compactum, $\dim X \leq n$. Denote by I the unit interval on the line.

Corollary 1.1. *The space X is locally compact if and only if the space $X \times I$ is peripherally compact.**

Now let X be a G_δ-space**. We shall say** ([2,3]) **that the space X is a G_δ-space of the** first kind** if there exists a compact extension cX such that

$$X = G_1 \cap G_2 \cap \cdots,$$

where the sets G_i are open in cX and satisfy the condition

$$\dim(cX \setminus G_i) < \dim X$$

for all $i = 1, 2, \ldots$. The space X is called a G_δ-space of the **second kind** if X is not a G_δ-space of the first kind.

Corollary 1.2. *If the space X is compact and finite-dimensional, whereas the G_δ-space Y is not locally compact and is zero-dimensional, then $X \times Y$ is a G_δ-space of the second kind.*

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* The space X is called **peripherally compact at the point** $x \in X$ if the point x has arbitrarily small neighborhoods in X whose boundaries are compact. The set of all points of the space X at which X is peripherally compact will be denoted by $P(X)$. Clearly, $P(X)$ is a subset of type G_δ in X. If $X = P(X)$, then X is called **peripherally compact**. By a well-known theorem of Freudenthal ([1]), the space X is peripherally compact if and only if there exists a compact extension cX for it such that the remainder $cX \setminus X$ is zero-dimensional.

** A space is called a G_δ-space if it is a subset of type G_δ in some compact extension of its own, or—what is the same thing—if it is homeomorphic to some complete space.

Corollaries 1 and 2 are a generalization of the theorem from ([3]), where it was required that the space Y be lacunar*. From Corollaries 1 and 2 there follows directly an example giving a negative answer to one of the questions posed in ([3]) (P313). Namely, it will be the space $I \times Y^0$, where $Y^0 = \{0\} \cup \{1/2^i + 1/3^j;\ i, j = 1, 2, \ldots\}$.

Another contradictory example, not being, as here, the sum of a countable number of parallel intervals, was given by Reichaw-Reichbach ([4]). Similarly, the product $Q^n = I^n \times Y^0$ is a G_δ-space of the second kind—it is the sum of a sequence of disjoint cubes. Every compact extension of the space Q^n has at least an n-dimensional growth. The following example of Reichaw-Reichbach ([5]) has the same property:

$$R^n = (\operatorname{fr} I^n \times \{0\}) \cup \left(I^n \times \left\{ \frac{1}{i} :\ i = 1, 2, \ldots \right\} \right),$$

where $\operatorname{fr} I^n$ is the boundary of the cube I^n in n-dimensional Euclidean space. Let $\operatorname{def} X$ denote the defect of the space X, i.e. the least dimension of the growth under compact extensions of the space X. Then $\operatorname{def} Q^n = \operatorname{def} R^n = n$. Moreover, for the space R^n we have $\dim(R^n \setminus P(R^n)) = n-1$. We shall see (cf. Corollary 2.2) that the dimension of the latter set cannot be lowered.

Define the number $\operatorname{Com} X$ analogously to the large inductive dimension $\operatorname{Ind} X$, but starting from compact spaces instead of the empty space. Namely, $\operatorname{Com} X = -1$ if and only if X is compact. The inequality $\operatorname{Com} X \leq n$ is valid if, for every closed subset A of the space X and every neighborhood OA of it, there exists a neighborhood $O'A$, lying in OA, such that $\operatorname{Com} \operatorname{fr} O'A \leq n-1$. From the examples of the spaces Q^n and R^n it is clear that, for the number $\operatorname{Com} X$, the sum theorem is, generally speaking, not true. It turns out that the number $\operatorname{Com} X$ can be used for an internal characterization of the defect $\operatorname{def} X$. Namely, as was already noted by Grote, the inequality $\operatorname{Com} X \leq \operatorname{def} X$ is easily proved by induction (see ([6]), p. 178) with respect to the defect $\operatorname{def} X$. On the other hand, the inequality $\operatorname{def} X \leq \operatorname{Com} X$ follows from the work of Vries ([7]). Thus we have the formula $\operatorname{def} X = \operatorname{Com} X$, which we shall call the **Vries-Grote formula**.

Theorem 2. *For every space X the inequality holds*

$$\operatorname{Com} X \leq \dim(X \setminus P(X)) + 1.$$

Proof. From Freudenthal' s theorem ([1]) it follows that a peripherally compact space X satisfies the condition $\operatorname{Com} X \leq 0$. If A is a closed subset of some space X, then $A \cap P(X) \subseteq P(A)$. Hence, by induction ([6]) with respect to the dimension of the set $X \setminus P(X)$, we obtain the assertion of Theorem 2.

Let us note that the dimension of the set $X \setminus P(X)$ may exceed the quantity $\operatorname{Com} X - 1$. This property is possessed, for example, by the above-mentioned space Q^n, for which, by the Vries-Grote formula, $\operatorname{Com} Q^n = n$. But $\dim(Q^n \setminus P(Q^n)) = n$. Consequently, the inequality in Theorem 2 cannot be replaced by an equality.

Corollary 2.1. *For every space X there exists a compact extension dX such that*

$$\dim(dX \setminus X) \leq \dim(X \setminus P(X)) + 1.$$

Corollary 2.1 is obtained with the aid of the Vries-Grote formula. It may be understood as a generalization of Freudenthal' s theorem ([1]).

Corollary 2.2. *If the G_δ-space X is finite-dimensional and*

$$\dim(X \setminus P(X)) \leq \dim X - 2,$$

then X is a Q_δ-space of the first kind.

We now pass to infinite-dimensional spaces. We shall call a space Z a **pseudopolyhedron** if it admits a representation in the form of a sum $Z = \Sigma_1 \cup \Sigma_2 \cup \cdots$ of a countable system $\{\Sigma_i\}$ of simplexes

* We call a space **lacunar** if it is not locally compact at any of its points, or, equivalently, if all its compact subsets are nowhere dense.

of any dimension* such that for all values $i, j = 1, 2, ...$ the intersection $\Sigma_i \cap \Sigma_j$ is a face of both complexes Σ_i and Σ_j, and the diameter of the simplex Σ_i tends to zero as i increases without bound. If the system $\{\Sigma_i\}$ is locally finite, then Z is a polyhedron (infinite and hence, generally speaking, noncompact).

Lemma. *If X is a closed subset of a compact space Y and $\varepsilon > 0$, then there exists an ε-mapping f of the space Y into a Hilbert brick such that*

$$f(X) \cap f(Y \setminus X) = \emptyset,$$

$f|X$ is a homeomorphism, and $f(Y \setminus X)$ is a polyhedron.

The proof of the lemma is obtained with the aid of the known theorems on mappings of spaces into nerves of their open and, in the present case, countable covers (see ([6]), pp. 114 and 212–217).

Theorem 3. *For any G_δ-space X and its compact extension cX there exists a compact extension eX such that cX maps onto eX and the growth $eX \setminus X$ is a pseudopolyhedron.*

Proof. We have $cX \setminus X = F_1 \cup F_2 \cup ...$, where the F_i are compact sets. Put $Y_0 = \emptyset$. According to the lemma, there exists a $(1/i)$-mapping f_i of the compact set $Y_i = F_1 \cup ... \cup F_i$ such that

$$f_i(Y_{i-1}) \cap f_i(Y_i \setminus Y_{i-1}) = \emptyset,$$

$f_i|Y_{i-1}$ is a homeomorphism, and $f_i(Y_i \setminus Y_{i-1})$ is a polyhedron $(i = 1, 2, ...)$. Consider the decomposition of the compact set cX into sets of the form $f_i^{-1}(z)$ for $z \in f_i(Y_i \setminus Y_{i-1})$ and into the points $x \in X$. This decomposition induces a certain mapping f of the compact set cX. The image $f(cX)$ is the required extension eX.

Corollary 3.1. *All infinite-dimensional G_δ-spaces are G_δ-spaces of the first kind.*

Corollary 3.1 gives a negative answer to question (P 312) of the paper ([3]).

Corollary 3.2. *For any G_δ-space there exists a compact extension with weakly countable-dimensional** growth.*

Let us call a **complete extension** of a space X any G_δ-space containing a homeomorphic image of the space X as a dense subset. Of course, every compact extension of a given space is its complete extension.

Corollary 3.3. *A compact extension of a given space such that it or its growth is countable-dimensional** or weakly infinite-dimensional**** exists if and only if there exists, respectively, such a complete extension of the given space.*

In the case of weakly infinite-dimensional extensions this follows from the well-known theorem of Levshenko ([8]) and the preceding Corollary 3.2.

Let H denote the subset of the Hilbert brick $I^{\aleph_0}$ consisting of all those points at which only a finite number of coordinates are nonzero. As Sklyarenko proved ([9]), every compact extension of the space H is strongly infinite-dimensional. Thus ([8]), the space H, being a weakly countable-dimensional F_δ-space***** and even a pseudopolyhedron, does not admit a compact extension with weakly infinite-dimensional growth. This shows that the condition of being a G_δ-space is essential in Corollary 3.2. Further, by Corollary 3.3, every complete extension of the space H is strongly infinite-dimensional and has strongly infin-

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* Curves are considered (i.e. homeomorphic to ordinary) closed simplices. A (−1)-dimensional simplex is understood as the empty set.

** A space is called **weakly countable-dimensional** if it is the sum of a countable system of finite-dimensional subsets closed in it.

*** A space is called **countable-dimensional** if it is the sum of a countable system of zero-dimensional subsets.

**** A space X is called **weakly infinite-dimensional** if, for any countable system of pairs of closed subsets A_i, B_i such that

$$A_i \cap B_i = \emptyset \quad (i = 1, 2, ...),$$

there exist closed subsets C_i such that C_i separates X between A_i and B_i $(i = 1, 2, ...)$, and the intersection of all the sets C_i is empty. In the opposite case the space X is called **strongly infinite-dimensional**.

***** A space is called an F_δ-space if it is the sum of a countable system of compact subsets.

a nonmetrizable remainder. It follows from this that Tumarkin' s theorem [(10)] on the existence, for finite-dimensional sets, of their "supersets" of type G_δ of the same dimension cannot be extended to infinite-dimensional sets.

Another F_σ-space, namely the space $I^{\aleph_0} \times W$, where W is the space of all rational numbers, is all the more similar to the finite-dimensional space Q^n considered above in that, like Q^n, it is the product of a compactum and a zero-dimensional set. At the same time it has a certain property of the space H, which is ensured by the following theorem, representing one possible analogue of Theorem 1 for infinite-dimensional spaces.

Theorem 4. *If the space X is locally compact, the F_σ-space Y is lacunary, and there exists a compact extension $c(X \times Y)$ such that the remainder $c(X \times Y) \setminus (X \times Y)$ is weakly countable-dimensional, then the space X contains a finite-dimensional nonempty open subset.*

Proof. Let

$$M = c(X \times Y) \setminus (X \times Y).$$

The remainder M is weakly countable-dimensional, so that there is a decomposition of it into a sum

$$M = M_1 \cup M_2 \cup \cdots$$

of finite-dimensional subsets M_i $(i = 1, 2, ...)$ closed in M. Since X, like Y, is an F_σ-space, the same is true of their product $X \times Y$, and therefore the set M is a G_δ-space. Thus, among the sets M_i there is a set M_k which contains some nonempty open subset G of the space M. Consider an open subset G' of the compactum $c(X \times Y)$ such that $G = G' \cap M$, and take a nonempty open set G'' in $c(X \times Y)$, whose closure is contained in G'. Let X'' and Y'' be such nonempty open subsets of the spaces X and Y, respectively, whose product $X'' \times Y''$ lies in $G'' \cap (X \times Y)$. Since the space X is locally compact, there exists a compact set $X' \subseteq X''$ containing a nonempty open subset U of the space X. We shall show that U is finite-dimensional.

Indeed, choose some nonempty set $Y' \subseteq Y''$ which is the closure of an open set in Y. Since the space Y is lacunary, the space Y' is not locally compact. Denote by

$$c'(X' \times Y')$$

the closure of the set $X' \times Y'$ in the compactum $c(X \times Y)$. Thus, $c'(X' \times Y')$ is a compact extension of the space $X' \times Y'$. From Theorem 1 it follows that the dimension of the space X' does not exceed the dimension of the remainder M' in this extension. But the set $X' \times Y'$ is closed in $X \times Y$, and therefore $M' \subseteq M$. On the other hand, the remainder M' lies in the closure of the set $X' \times Y' \subseteq X'' \times Y'' \subseteq G''$. Hence,

$$M' \subseteq G' \cap M = G \subseteq M_k.$$

It follows that

$$\dim U \leq \dim X' \leq \dim M' \leq \dim M_k.$$

Remark. It follows from Theorem 4 that the space $I^{\aleph_0} \times W$ admits no compact extension with weakly countable-dimensional remainder. It seems interesting to study the dimensional properties of the remainders of the space $I^{\aleph_0} \times W$ under compact extensions. For example, is the remainder of an arbitrary compact extension of the space $I^{\aleph_0} \times W$ strongly infinite-dimensional? Also unresolved remains the following question, connected with Corollaries 3.2 and 3.3: does every weakly countable-dimensional G_δ-space have a weakly countable-dimensional compact extension?

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Note: Figure translations are in progress. See original paper for figures.