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Abstract

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PERIODS OF $\mathfrak{A}$-CLOSED FORMS

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In this paper the notion of periods of $\mathfrak{A}$-closed forms with respect to cycles of a manifold is introduced. Results generalizing the well-known theorems of de Rham and Hodge (see ([3–6])) are established in terms of periods. The concept of a period is of important significance in a number of questions connected with A-harmonic fields and forms, in particular in the consideration of A-harmonic fields with singularities.

1. Let $\mathfrak{M}$ be a real analytic compact Riemannian space of dimension n (without boundary), on which analytic tensors $a_j^{i_1,\dots,i_q}$ $(q = 0, \dots, m)$, contravariant in the upper indices and covariant in the lower one, are given. We define operators $\mathfrak{A}^p$ $(p = 0, \dots, n-1)$, taking forms of degree p into forms of degree $p+1$, by the formula

$$(\mathfrak{A}^p\alpha)_{k_1,\dots,k_{p+1}} = \sum_{\nu=1}^{p+1}(-1)^{\nu-1}A_{k_\nu}\alpha_{k_1,\dots,\hat{k}_\nu,\dots,k_{p+1}}; \qquad A_j = \sum_{q=0}^{m} a_j^{i_1,\dots,i_q}\nabla_{i_1}\cdots\nabla_{i_q}.$$

Here ∇_i is the symbol of covariant differentiation. The indices on the forms $\alpha, \mathfrak{A}^p_\alpha$ indicate that the corresponding components of the forms are meant; $k_1, \dots, \hat{k}_\nu, \dots, k_{p+1}$ is obtained from $k_1, \dots, k_{p+1}$ by deleting k_ν. In A_j summation over repeated indices is carried out. We also define

$$A^p = (\mathfrak{A}^p)'\mathfrak{A}^p + \mathfrak{A}^{p-1}(\mathfrak{A}^{p-1})',$$

where $(\mathfrak{A}^p)'$ is metrically adjoint to $\mathfrak{A}^p$ (see ([4])). It is assumed that A^0 is an elliptic operator and the A_j commute.

A form α is called $\mathfrak{A}$-closed if $\mathfrak{A}\alpha = 0$; β is called $\mathfrak{A}$-homologous to zero if there exists on $\mathfrak{M}$ such a form γ that $\mathfrak{A}\gamma = \beta$. A form φ is called A-harmonic if $A\varphi = 0$; ψ is called an A-harmonic field if $\mathfrak{A}\psi = 0$, $\mathfrak{A}'\psi = 0$.

The following generalized de Rham theorem holds.

Theorem 1. *The quotient group $\mathcal{H}^p_{\mathfrak{A}}(\mathfrak{M})$ of the group of $\mathfrak{A}$-closed forms of degree p by the subgroup of $\mathfrak{A}$-homologous-to-zero forms of degree p is isomorphic* *to $[H^p(\mathfrak{M}, R)]^M$, where $H^p(\mathfrak{M}, R)$ is the p-dimensional cohomology group of the space $\mathfrak{M}$ with real coefficients. M is the number of linearly independent $\mathfrak{A}$-closed forms of degree zero.*

2. Here forms $\beta_p(u, \tilde{v}_j)$, fundamental for what follows, will be introduced and an assumption concerning them will be formulated.

Let

$$\mathfrak{A}^{*p} = *(\mathfrak{A}^{n-p})' *^{-1}$$

(with respect to the operator $*$, see ([4])). On $\mathfrak{M}$ there exist M linearly independent solutions of the equation $\mathfrak{A}^{*0} v = 0$, and also of the equation $\mathfrak{A}^0 u = 0$. We denote these solutions respectively by

$$\tilde{v}_1, \dots, \tilde{v}_M; \quad \tilde{u}_1, \dots, \tilde{u}_M.$$

Lemma 1. *There exist forms $\beta_p(u, \tilde{v}_j)$ $(1 \le j \le M)$ of degree p $(0 \le p \le n)$ such that for all u of degree p the relations hold*

$$\beta_n(u, \tilde{v}_j) = u \cdot \tilde{v}_j; \qquad \beta_{p+1}(\mathfrak{A}^p u, \tilde{v}_j) = d\beta_p(u, \tilde{v}_j) \quad (p < n).$$

Here the coefficients $\beta_p(\tilde{u}, v_j)$ are bilinear differential expressions with respect to u, $\tilde{v}_j$ (d is the exterior differential).

Let $S_{\varepsilon_1, \varepsilon_2} :\ \varepsilon_1 < r_{P,Q} < \varepsilon_2$ be a geodesic ring with center at an arbitrary point $P \in \mathfrak{M}$. By Theorem 1 the group $H^{n-1}_{\mathfrak{A}}(S_{\varepsilon_1, \varepsilon_2})$ has M generators $\hat{u}_1, \dots, \hat{u}_M$. We shall assume that

$$\det \|B_{ij}\| \ne 0, \qquad (\beta_{i,j})_{j=1}^{M} \ne (0, \dots, 0),$$

where $B_{i,j}$ are given by the formula

$$\int_{r_{P,Q} = \varepsilon} \beta_{n-1}(u_i, \tilde{v}_j) \qquad (\varepsilon_1 < \varepsilon < \varepsilon_2)$$

and do not depend on ε, while $\beta_{i,j}$ are numbers on $\mathfrak{M}$ equal to $\beta_0(\tilde{u}_i, \tilde{v}_j)$.

3. Let K be some simplicial decomposition (see ([3])) of the manifold $\mathfrak{M}$. Introduce the mapping B_i of the set of all forms of degree p on $\mathfrak{M}$ into the set of chains of the complex K:

$$B_i(\varphi^p) = \sum_{C^p} \{\varphi^p, C^p\}_i \, C^p; \qquad \{\varphi^p, C^p\}_j = \int_{C^p} \beta_p(\varphi^p, \tilde{v}_j).$$

The following theorem is of fundamental importance:

Theorem 2. The operator $B\varphi = (B_i\varphi)_{i=1}^{M}$ establishes the isomorphism formulated in Theorem 1.

It is now possible to introduce

Definition. The **periods** of an $\mathfrak{A}$-closed form φ with respect to a cycle Z of the manifold $\mathfrak{M}$ are the numbers $\{\varphi, Z\}_i,\ 1 \leq i \leq M$.

4. On the basis of the results of item 3, the following theorems are established:

Theorem 3. An $\mathfrak{A}$-closed form with zero periods with respect to all cycles of the manifold is $\mathfrak{A}$-homologous to zero.

Let Z_j^p $(j = 1, \dots, s)$ be p-dimensional cycles no linear combination of which is homologous to zero. Then:

Theorem 4. There exists an $\mathfrak{A}$-closed form having arbitrary prescribed periods with respect to the cycles Z_j^p.

Theorem 5. There exists an A-harmonic form having arbitrary prescribed periods with respect to the cycles Z_j^p. The latter is uniquely determined if s is equal to the p-th Betti number.

In the case of the operator $\mathfrak{A}$ coinciding with the operator of exterior differentiation, Theorems 3 and 4 were obtained by de Rham ([5]), and Theorem 5 is known as Hodge' s theorem ([4,6]).

5. In conclusion we indicate analogues of Theorems 3 and 4 for the case of differential forms with fixed carrier. These results are needed in the consideration of A-harmonic fields and forms on a manifold with boundary (cf. ([1,2])).

We shall denote the carrier of the form φ by $\underline{\varphi}$. Let $\mathfrak{N}$ be an n-dimensional subspace of $\mathfrak{M}$ with boundary $\mathfrak{B}$, and let R_i^p be a basis of the relative p-dimensional cycles of $\mathfrak{N}$ (mod $\mathfrak{B}$). The **relative periods** of the form φ $(\underline{\varphi} \subseteq \mathfrak{N})$ are the numbers $\{\varphi, R_i^p\}_j$.

Theorem 3′. An $\mathfrak{A}$-closed form φ $(\underline{\varphi} \subseteq \mathfrak{N})$ with zero relative periods is representable in the form $\varphi = \mathfrak{A}\psi$ $(\underline{\psi} \subseteq \mathfrak{N})$.

Theorem 4′. There exists an $\mathfrak{A}$-closed form θ $(\underline{\theta} \subseteq \mathfrak{N})$ having arbitrary prescribed relative periods.

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References

1. G. F. D. Duff, Ann. Math., **56**, 1, 115 (1952).

2. G. F. D. Duff, Spencer, Ann. Math., **56**, 1, 128 (1952).

3. K. Kodaira, Ann. Math., **50**, 3, 587 (1949).

4. G. de Rham, *Differentiable Manifolds*, IL, 1956.

5. G. de Rham, J. math. pures et appl., **10**, 115 (1931).

6. W. V. D. Hodge, Proc. London Math. Soc., **44**, 483 (1936).

Note: Figure translations are in progress. See original paper for figures.