CONDITIONS FOR DISCRETENESS AND CONTINUITY OF THE SPECTRUM IN THE CASE OF A SELF-ADJOINT SYSTEM OF FIRST-ORDER DIFFERENTIAL EQUATIONS

MATHEMATICS

1965

SovietRxiv

Abstract

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UDC 517.948.35:513.88:517.925

MATHEMATICS

V. V. MARTYNOV

CONDITIONS FOR DISCRETENESS AND CONTINUITY OF THE SPECTRUM IN THE CASE OF A SELF-ADJOINT SYSTEM OF FIRST-ORDER DIFFERENTIAL EQUATIONS

(Presented by Academician I. M. Vinogradov on 16 VI 1965)

Let the operator D be generated by the differential expression

$$Dy = i\Lambda y' + Q(x)y \tag{1}$$

in the space $L_2(-\infty, +\infty)$ of vector-functions $y(x) = (y_1, \ldots, y_k)$ with scalar product

$$[a, b] = \int_{-\infty}^{+\infty} (a, b)\, dx = \int_{-\infty}^{+\infty} \sum_{m=1}^{k} a_n(x)\overline{b_n(x)}\, dx.$$

It is assumed that in the principal part of the operator there stands a constant real diagonal invertible matrix, and that the matrix-function $Q(x) \equiv Q^*(x)$ is continuous. Under these conditions the operator D is self-adjoint and unbounded in both directions.

If the dimension of the unitary space E_k is even ($k = 2p$) and

$$\Lambda = \begin{pmatrix} 1_p & 0_p \\ 0_p & -1_p \end{pmatrix},$$

then, in essence, we are dealing with a Hamiltonian, i.e., with the operator

$$T = J\frac{d}{dx} + R(x) = UDU^{-1} \quad \text{for} \quad J = \begin{pmatrix} 0_p & 1_p \\ -1_p & 0_p \end{pmatrix}, \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \cdot 1_p & i \cdot 1_p \\ 1_p & 1_p \end{pmatrix}, \tag{2}$$

which arises from the canonical system of equations $J(dy/dx) = H(x, \lambda)y$ in the case when the spectral parameter λ enters additively into the Hermitian matrix

$H(x,\lambda)$. For $p=1$ the operator T was studied in a number of papers by E. C. Titchmarsh on the relativistic Dirac equation.

The main content of this note is the assertion that the operator D may have a purely discrete spectrum. This circumstance is specific to the case of a system of equations, since in the case of a single equation the spectrum fills the entire axis continuously (and thus the system (1) with diagonal $Q(x)$ has a continuous spectrum on the whole axis, since then D decomposes into k copies of scalar operators).

We first consider a special case which is of independent interest. Let $\Lambda^2 = 1_k$ and suppose that $Q'(x)$ exists. Then the operator D^2 is unitarily equivalent to an operator of Sturm-Liouville type. Indeed,

$$D^2 = -d^2/dx^2 + i(\Lambda Q + Q\Lambda)d/dx + (Q^2 + i\Lambda Q'),$$

since

$$(d/dx)Q = Q' + Q(d/dx).$$

Next,

$$A^{-1}D^2A = -d^2/dx^2 + A^{-1}[-2A' + i(\Lambda Q + Q\Lambda)A]\,d/dx + A^{-1}[-A'' + i(\Lambda Q + Q\Lambda)A' + (Q^2 + i\Lambda Q')A]$$

for any invertible twice differentiable matrix $A(x)$. The equation

$$A'(x) = (i/2)[\Lambda Q(x) + Q(x)\Lambda]A(x)$$

always has a unitary solution $U(x)$ (for example, the solution with $U(0) = 1_k$); therefore, finally, we obtain

$$U^{-1}D^2U = -d^2/dx^2 + U^{-1}(x)X(x)U(x),$$

where

$$X(x) = Q^2(x) - \frac{1}{4}(\Lambda Q + Q\Lambda)^2 + \frac{i}{2}(\Lambda Q' - Q'\Lambda).$$

Thus, we can conclude: if

$\Lambda = \pm 1_k$, then the spectrum of the operator D continuously fills the entire axis. If, however, $\Lambda \neq \pm 1_k$ (although $\Lambda^2 = 1_k$), then condition $*$

$$\lim_{|x|\to\infty} \mu X(x) = +\infty \tag{3}$$

guarantees the discreteness of the spectrum of the operator D.

In Theorem 1 a method is proposed for obtaining sufficient criteria for discreteness of the spectrum.

Theorem 1. 1) If, for some differentiable matrix $B(x) \equiv B^*(x)$,

$$\lim_{|x|\to\infty} \mu \left[B' - B\Lambda^{-2}B - 2\,\mathrm{Im}(B\Lambda^{-1}Q)\right] = +\infty, \tag{4}$$

then the operator D has discrete spectrum. For example, the spectrum is discrete if

$$\lim_{|x|\to\infty} \nu \left[\mathrm{Im}(B_0\Lambda^{-1}Q(x))\right] = -\infty \tag{5}$$

for some constant matrix $B_0 = B_0^*$.

2) On the other hand, if the spectrum is discrete, then necessarily the matrix $Q(x)$ is nondiagonal and satisfies the condition

$$\lim_{|x|\to\infty} \mu \int_x^{x+\omega} Q^2(t)\,dt = +\infty \quad (\omega \text{ fixed}), \tag{6}$$

and in the series of numbers $\lambda_1, \dots, \lambda_k$ standing on the diagonal of the matrix Λ, there must be at least one change of sign.

Let us illustrate all that has been said by the example of the operator

$$D_0 = i \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} p & q \\ \bar{q} & r \end{pmatrix}, \tag{7}$$

where $\lambda_1\lambda_2 \neq 0$, $q(x) = R(x) + iI(x)$. Let $\lambda_2 = -\lambda_1 = -1$ and let p', r', q' exist. Then condition (3) is rewritten in the following form:

$$\lim_{|x|\to\infty} \left[|q(x)|^2 - |q(p+r) + iq'|\right] = +\infty. \tag{8}$$

We give several sufficient criteria for discreteness of the spectrum of the operator D_0, obtained with the aid of Theorem 1:

1) If $\lambda_2 = -\lambda_1$, then from (5) one can obtain the criterion

$$\lim_{|x|\to\infty}\left[\alpha R(x)+\beta I(x)-\frac{1}{2}|p(x)+r(x)|\right]=+\infty \quad (9)$$

for some α and β, where $\alpha^2+\beta^2=1$. In the case of arbitrary λ_1, λ_2 this condition becomes less transparent:

$$\lim_{|x|\to\infty}\left[(\lambda_1-\lambda_2)(\alpha R+\beta I)-\sqrt{(\lambda_1+\lambda_2)^2(\alpha R+\beta I)^2+(p\lambda_2-r\lambda_1)^2}\right]=+\infty \quad (10)$$

for some α, β ($\alpha^2+\beta^2=1$).

2) If $R'(x)$ exists, then the condition

$$\lim_{|x|\to\infty}\left[-2\lambda_1\lambda_2 R^2(x)-|\lambda_1-\lambda_2|\,|R(p\lambda_2-r\lambda_1)+i\lambda_1\lambda_2 R'|\right]=+\infty \quad (11)$$

is sufficient for discreteness of the spectrum of the operator D_0. For $\lambda_2=-\lambda_1$ one can additionally obtain the condition

$$\lim_{|x|\to\infty}\left[\pm\sqrt{2}\,R(x)I(x)-|R(p+r)+i\lambda_1 R'|\right]=+\infty. \quad (12)$$

We note that the last two assertions remain valid if in them one formally replaces $R(x)$ by $I(x)$, and $I(x)$ by $R(x)$, since the passage from D_0 to the operator UD_0U^{-1} with the unitary matrix

$$U=\begin{pmatrix}0 & \exp(ia_0)\\ \exp(ib_0) & 0\end{pmatrix}$$

can interchange, in particular, $R(x)$ and $I(x)$.

* μN, νN and $\|N\|$ will denote, respectively, the least and greatest proper values and the Euclidean norm of

3) If $R'(x)$ and $I'(x)$ exist simultaneously, then each of the following two conditions guarantees discreteness of the spectrum:

$$\lim_{|x|\to\infty}\left[-\sqrt{2\lambda_1\lambda_2}(R\pm I)^2-|\lambda_1-\lambda_2|\left((R\pm I)(p\lambda_2-r\lambda_1)+i\lambda_1\lambda_2(R'\pm I')\right)\right]=+\infty, \quad (13)$$

$$\lim_{|x|\to\infty}\left[-2\lambda_1\lambda_2|q(x)|^2-|\lambda_1-\lambda_2|\,|q(\lambda_2 p-\lambda_1 r)+i\lambda_1\lambda_2 q'|\right]=+\infty. \quad (14)$$

In particular, from (14), for $\lambda_2 = -\lambda_1 = -1$, condition (8) is obtained, and without the requirement of differentiability of $p(x)$ and $r(x)$.

Proof of Theorem 1. Using the splitting principle ((1), §§ 1, 2), it is not hard to show that the spectrum of the operator D is discrete if and only if $\lim_{r\to+\infty} d(r) = +\infty$, where

$$d(r) = \inf_{y\in K(r,\infty)} \frac{\|Dy\|^2}{\|y\|^2} = \inf_y \frac{\displaystyle\int_{|x|\geq r} \left[|\Lambda y'|^2 + |Qy|^2 - 2\operatorname{Re}(iQy, \Lambda y')\right] dx}{\displaystyle\int_{|x|\geq r} |y|^2\, dx}. \tag{15}$$

Here $K(r,\infty)$ denotes the class of finite piecewise-smooth vector functions, the compact support of each of which is situated outside the interval $(-r,+r)$. Since always

$$\int_a^b (\Phi' f, g)\, dx = (\Phi f, g)\Big|_a^b - \int_a^b (\Phi f', g)\, dx - \int_a^b (\Phi f, g')\, dx,$$

then for any $y(x) \in K(r,\infty)$ and any self-adjoint differentiable matrix $B(x)$,

$$0 = \int_{|x|\geq r} \left[(B'y, y) + 2\operatorname{Re}(By, y')\right] dx. \tag{16}$$

Add expression (16) to the numerator in (15), and estimate the result from below as follows:

$$\int_{|x|\geq r} |Dy|^2\, dx = \int_{|x|\geq r} \left\{|\Lambda y'|^2 + (Q^2 y, y) + (B'y, y) + 2\operatorname{Re}(Py, \Lambda y')\right\} dx \geq$$

$$\geq \int_{|x|\geq r} (Cy, y)\, dx \geq \left[\inf_{|x|\geq r} \mu C(x)\right] \int_{|x|\geq r} |y|^2\, dx,$$

where

$$P(x) = \Lambda^{-1}B - iQ; \qquad C(x) = Q^2 + B' - P^*P = B' - B\Lambda^{-2}B - 2\operatorname{Im}(B\Lambda^{-1}Q).$$

In this estimate we have applied the Cauchy-Bunyakovsky inequality:

$$2\operatorname{Re}(Py, \Lambda y') \geq -2|Py|\,|\Lambda y'| \geq -|\Lambda y'|^2 - |Py|^2.$$

Condition (4) is proved.

Assertion (6) is a very particular case of the general fact which is formulated here for the one-dimensional case.

Lemma. Consider in $L_2(-\infty, +\infty)$ the operator

$$L = \sum_{i=0}^{l} Q_i(x) \frac{d^i}{dx^i},$$

where the matrices $Q_i(x)$ are continuous. If its resolvent R_λ is completely continuous, then

$$\lim_{|x|\to\infty} \mu \int_x^{x+\omega} \sum_i Q_i^* Q_i \, dt = +\infty \qquad (\omega \text{ fixed}). \tag{17}$$

Remark. If among the Q_i constant matrices occur, then one may suppose that they do not enter under the summation sign in (17), since the inequality

$$\mu A_0 + \mu B(x) \leq \mu[A_0 + B(x)] \leq \nu A_0 + \mu B(x)$$

is valid.

For the proof of the lemma assume that (17) is not fulfilled. Then there exist numbers $|x_s| \to \infty$ and a sequence of constant vectors $|\xi_s| = 1$, for which

$$\int_{x_s}^{x_s+\omega} \sum_i (Q_i^* Q_i \xi_s, \xi_s) \, dt < C_0$$

for some ···

$\omega_0 > 0,\ c_0 > 0$. Fix an arbitrary smooth function $u_0(x)$, equal to zero outside $[0, \omega_0]$. Then the sequence of vector-functions $y_s(x) = u_0(x - x_s)\xi_s$ is noncompact in $L_2(-\infty, +\infty)$, whereas

$$\|Ly_s - z_0 y_s\|^2 \leqslant 2\|z_0 y_s\|^2 + C_1 \sum_i \|Q_i y_s^{(i)}\|^2 =$$

$$= C_2 + C_1 \sum_i \int_{x_s}^{x_s+\omega_0} |u_0^{(i)}|^2 |Q_i \xi_s|^2 \, dt \leqslant C_2 + C_3 \sum_i \int_{x_s}^{x_s+\omega_0} |Q_i \xi_s|^2 \, dt \leqslant C_4,$$

which contradicts the complete continuity of the operator R_{z_0}. The lemma is proved.

Suppose that the initial operator D has discrete spectrum. "Normalize" the matrix Λ, i.e., pass from D to the operator $\widetilde{D} = i\widetilde{\Lambda}\, d/dx + \widetilde{Q}(x) = |\Lambda|^{-1/2} D |\Lambda|^{-1/2}$, for which $\widetilde{\Lambda}^2 = 1_k$ and $\widetilde{Q}^* \equiv \widetilde{Q}$. In this case the spectrum remains discrete, since the equation $\widetilde{D}y = \lambda y$ can be rewritten in the form $i\Lambda z' + Qz = \lambda|\Lambda|z$ for $z = |\Lambda|^{-1/2}y$, which corresponds to considering the operator D in the metric

$$\int_{-\infty}^{+\infty} (|\Lambda|y, y)\, dx,$$

and it is equivalent to the original one. Since the diagonal entries of the matrix $\widetilde{\Lambda}$ are the numbers $\lambda_i/|\lambda_i|$, there can be only two possibilities: either among the numbers $\Lambda_1, \dots, \Lambda_k$ there is at least one change of sign, or $\widetilde{\Lambda} = \pm 1_k$. Since from the very beginning we could assume $Q(x)$ differentiable (the discreteness of the spectrum, by virtue of (15), is not destroyed by arbitrary additive bounded perturbations of the matrix $Q(x)$), it follows that for $\widetilde{\Lambda} = \pm 1_k$ the spectrum is continuous on the whole axis, independently of the behavior of $\widetilde{Q}(x)$, as was shown above. Theorem 1 is completely proved.

We now give several conditions guaranteeing the continuity of the spectrum independently of the properties of the matrix Λ.

Theorem 2. 1) *The spectrum of the operator D is continuous on the whole axis if at least one of the following three conditions is satisfied: for every $\delta > 0$*

$$\int_{M_\delta} \|Q(t)\|^2\, dt < \infty, \tag{18}$$

where

$$M_\delta = \{x : \|Q'(x)\| > \delta\};$$

on some sequence of intervals Δ_i of unboundedly increasing length $|\Delta_i|$

$$\lim_{i\to\infty} \frac{1}{|\Delta_i|} \int_{\Delta_i} \|Q(t)\|^2\, dt = 0; \tag{19}$$

for some (and hence for every) $\omega > 0$

$$\lim_{|x|\to\infty} \left\| \int_x^{x+\omega} Q^2(t)\, dt \right\| = 0. \tag{20}$$

2) *If one denotes*

$$\Omega = \limsup_{|x|\to\infty} \nu Q(x) - \liminf_{|x|\to\infty} \mu Q(x),$$

then the length of each gap in the continuous spectrum does not exceed Ω*.*

Theorem 2 is proved by the method of I. M. Glazman ([1]), §31; to obtain (20) one uses inequality (15) from ([2]).

In conclusion the author expresses his deep gratitude to his supervisors M. A. Naimark and R. S. Ismagilov.

Moscow
Physico-Technical Institute

Received
15 VI 1965

CITED LITERATURE

[1] I. M. Glazman, *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators*, 1963. [2] R. V. Martynov, *Differential Equations*, 1, No. 12 (1965).

Note: Figure translations are in progress. See original paper for figures.