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Abstract

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ON THE STABILITY OF SOLUTIONS OF THE CAUCHY PROBLEM FOR LINEAR DIFFERENTIAL EQUATIONS OF HYPERBOLIC TYPE

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Questions of stability for partial differential equations have been studied relatively little. In the works of M. A. Rutman (see, for example, (^{1})), the stability of linear systems "with a leading term" under Goursat boundary conditions was considered. In the present work, for the first time, as far as we know, the stability of solutions of the Cauchy problem for systems of hyperbolic type is considered.

Let the curve $y = \varphi(x)$ satisfy the following conditions: 1) $\varphi(x)$ is defined on (a, ∞), $-\infty \leq a < 0$; 2) $\varphi(0) = 0$; 3) $\varphi(x)$ is strictly decreasing on (a, ∞); 4) $\sup \varphi(x) = +\infty$, $\inf \varphi(x) \geq -\infty$; 5) $\varphi(x)$ has at each point a finite continuous derivative $\varphi'(x) \neq 0$.

Consider the boundary-value problem

$$\partial^2 z/\partial x\,\partial y - A\,\partial z/\partial x - B\,\partial z/\partial y - Cz = f(x,y); \qquad z|_{y=\varphi(x)} = p(x),$$
$$\partial z/\partial x\big|_{y=\varphi(x)} = q(x). \tag{1}$$

Here $z(x,y)$, $f(x,y)$, $p(x)$, $q(x)$ are vector-functions whose values belong to the complex Banach space E; moreover, $p(x)$ and $q(x)$ are defined and continuous on (a,∞); $p(x)$ has a finite derivative on (a,∞); $f(x,y)$ is defined and continuous in the domain $\Pi\{y \geq \varphi(x)\}$; A, B, C are linear bounded operators acting in E.

We shall call the boundary-value problem (1) **stable** in the domain Π if to every right-hand side $f(x,y)$ bounded in Π and initial data $p(x)$ and $q(x)$ bounded on (a,∞) there corresponds a solution bounded in Π.

Consider the operator-function

$$R_{\lambda,\mu} = (\lambda\mu I - \lambda A - \mu B - C)^{-1}$$

of the complex variables λ, μ. A point (λ, μ) at which the operator $R_{\lambda,\mu}$ exists and is bounded will be called **regular** for $R_{\lambda,\mu}$; any other point will be called **singular**. The set of all singular points of $R_{\lambda,\mu}$ will be denoted by σ. To each point $(\lambda,\mu) \in \sigma$ we assign the corresponding point $(\operatorname{Re}\lambda, \operatorname{Re}\mu)$ with real coordinates. The set of all such points will be denoted by K, and its closure by $\overline{K}$.

Theorem 1. *In order that the boundary-value problem (1) be stable in the domain* Π, *it is necessary and sufficient that no point of the set* $\overline{K}$ *belong to the domain* $\Pi_1\{x \geq 0,\ y \geq 0\}$*:*

$$\overline{K} \cap \Pi_1 = \emptyset. \tag{2}$$

We give the main points of the proof.

Consider the operators S_1, S_2:

$$S_1 z(x,y) = \int_{\varphi^{-1}(y)}^{x} z(s,y)\,ds, \qquad S_2 z(x,y) = \int_{\varphi(x)}^{y} z(x,t)\,dt.$$

Here $\varphi^{-1}(y)$ is the function inverse to $\varphi(x)$, $(x,y) \in \Pi$.

Denote by D the domain lying in the s,t-plane, $s \leq x,\ \varphi(x) \leq t \leq y$. It is easy to obtain:

$$S_1 S_2 z(x,y) = S_2 S_1 z(x,y) = \iint\limits_D z(s,t)\,ds\,dt.$$

The boundary-value problem (1) is equivalent to the operator equation

$$z(x,y) - S_2 A z(x,y) - S_1 B z(x,y) - S_1 S_2 C z(x,y) = \\ = S_1 S_2 f(x,y) + S_1 m(x,y) + (S_2 A + I) n(x,y), \tag{3}$$

where $m(x,y) = Bp(x) + q(x)$, $n(x,y) = p[\varphi^{-1}(y)]$.

From the easily verified formulas

$$(I - \lambda S_1)^{-1} z(x,y) = z(x,y) + \lambda \int_{\varphi^{-1}(y)}^{x} e^{\lambda(x-s)} z(s,y)\,dt,$$

$$(I - \mu S_2)^{-1} z(x,y) = z(x,y) + \mu \int_{\varphi(x)}^{y} e^{\mu(y-t)} z(x,t)\,dt \tag{4}$$

it follows that $(I - \lambda S_1)^{-1}$, $(I - \mu S_2)^{-1}$ are entire operator-functions of the complex variables λ, μ. Therefore, as shown in (^{2}), the solution of equation (3) can be written in the form

$$z = \frac{1}{(2\pi i)^2} \oint_{\gamma} \oint_{\delta} (I - \mu S_1)^{-1} (I - \mu S_2)^{-1} R_{\lambda,\mu} \{ S_1 S_2 f + S_1 m +$$

$$+ (S_2 A + I) n \} \, d\lambda \, d\mu. \tag{5}$$

The contours γ, δ here are such that, for λ lying on γ or outside γ, and μ lying on δ or outside δ, the point (λ, μ) is regular for $R_{\lambda,\mu}$.

Let condition (2) be satisfied. Then

$$\alpha = \sup_{\sigma} \min\{\operatorname{Re}\lambda, \operatorname{Re}\mu\} < 0,$$

and the contours γ, δ can be chosen to lie in the left open half-planes: as γ one may take an arc of the circle $|\lambda| = R$, $\operatorname{Re}\lambda < \alpha'$, of sufficiently large radius, cut off by the chord $\operatorname{Re}\lambda = \alpha'$, $|\operatorname{Im}\lambda| \leq \sqrt{R^2 - (\alpha_1)^2}$ $(\alpha' < \alpha)$, and as δ the same kind of contour in the μ-plane. From formulas (5) and (6), taking into account that

$$(I - \lambda S_j)^{-1} S_j = \frac{1}{\lambda} [(I - \lambda S_j)^{-1} - I] \qquad (j = 1, 2),$$

and assuming $f(x, y)$, $p(x)$, and $q(x)$ to be bounded in norm, it is easy to obtain that $z(x, y)$ is also bounded.

Suppose now that at least one point of $\overline{K}$ belongs to Π_1. Assume first that it lies inside Π_1. Then there exists at least one point of K lying inside Π_1. Denote the corresponding point of σ by (λ_0, μ_0), $\lambda_0 = \lambda_0' + \lambda_0'' i$, $\mu_0 = \mu_0' + \mu_0'' i$; by assumption, $\lambda_0' > 0$, $\mu_0' > 0$.

We use the lemma stated in (^{1}), p. 791. For brevity we restrict ourselves to the case: a) let f_0 be a vector from E, $\|f_0\| = 1$, such that

$$R_{\lambda,\mu} f_0 = \frac{1}{(\lambda - \lambda_0)(\mu - \mu_0)} f_0 - R_{\lambda,\mu} \left[\frac{P}{\lambda - \lambda_0} + \frac{Q}{\mu - \mu_0} \right] f_0; \tag{6}$$

here $P = \lambda_0 I - B$, $Q = \mu_0 I - A$.

Consider the function

$$\psi(x, y) = \begin{cases} xy \exp[-xy + i(\lambda_0'' x + \mu_0'' y)], & \text{if } (x, y) \in \Pi_1, \\ 0, & \text{if } (x, y) \in \Pi_1, \end{cases}$$

$\psi(x, y)$ is continuous and bounded in Π. Denote by $z_0(x, y)$ the solution of the boundary-value problem (1) corresponding to the initial data

$$p(x) \equiv q(x) \equiv 0 \tag{7}$$

and to the right-hand side $f(x, y) = \psi(x, y) f_0$. Let $(x, y) \in \Pi_1$. From formulas (5), (6) one can obtain

$$z_0(x, y) = v(x, y) f_0 - (I - \lambda_0 S_1)^{-1} S_1 z_1(x, y) - (I - \mu_0 S_2)^{-1} S_2 z_2(x, y), \tag{8}$$

where

$$v(x, y) = \int_0^x \int_0^y \exp[\lambda_0(x - s) + \mu_0(y - t)] \psi(s, t)\, ds\, dt;$$

$z_1(x, y)$ and $z_2(x, y)$ are solutions of the boundary-value problem (1) corresponding to the initial data (7) and to the right-hand sides, respectively, $\psi(x, y) P f_0$ and $\psi(x, y) Q f_0$.

Let $z_1(x, y)$ and $z_2(x, y)$ be bounded in Π:

$$C = \max\left\{\sup_{\Pi} \|z_1(x, y)\|,\ \sup_{\Pi} \|z_2(x, y)\|\right\} < \infty.$$

From (4) the estimates follow:

$$\left\|(I - \lambda_0 S_1)^{-1} S_1 z_1(x, y)\right\| < \frac{C}{\lambda_0'} \left(e^{\lambda_0' x} - 1\right),$$

$$\left\|(I - \mu_0 S_2)^{-1} S_2 z_2(x, y)\right\| < \frac{C}{\mu_0'} \left(e^{\mu_0' y} - 1\right).$$

On the other hand, it can be shown that for sufficiently large x

$$\left| v\left(x, \frac{\lambda_0'}{\mu_0'} x\right) \right| > C_1 e^{\lambda_0' x} \ln\left(1 + \frac{x}{\mu_0'}\right),$$

where $C_1 > 0$ is some constant. Since for $y = \dfrac{\lambda_0'}{\mu_0'} x$ one has $e^{\mu_0' y} = e^{\lambda_0' x}$, it follows from the estimates given and from (8) that the function $z_0(x, y)$ is unbounded on the ray

$$y = \frac{\lambda_0'}{\mu_0'} x, \qquad x > 0.$$

Thus, at least one of the functions $z_0(x,y), z_1(x,y), z_2(x,y)$ is unbounded in Π; since all of them are solutions of (1) with bounded right-hand sides and zero initial data, the problem is unstable.

The case when a point of K lies on the boundary of Π_1 reduces to the one considered, since if for certain coefficients A, B, C the boundary-value problem (1) is stable, then it is also stable for coefficients A', B', C' sufficiently close to them in norm.

Remark 1. In the simplest case, when A, B, C are real numbers, condition (2) is equivalent to the conditions

$$A < 0, \qquad B < 0, \qquad C < 0.$$

Remark 2. Let us continuously deform the curve $y = \varphi(x)$ so that, while conditions 1)–5) are preserved, it approaches the broken line consisting of the positive coordinate semiaxes. By a limiting passage one can obtain the stability criterion for the Goursat boundary-value problem, due to M. A. Rutman and formulated in (1) in other terms and for equations of a more general form.

We have considered the stability of solutions of the Cauchy problem (1) in the domain $\Pi\{y \geq \varphi(x)\}$. We give criteria for the stability of solutions of this problem in some other domains of variation of the independent variables.

Theorem 2. Let Π' be the domain $0 \geq y \geq \varphi(x)$, and let $\inf \varphi(x) = -\infty$. In order that the boundary-value problem (1) be stable in the domain Π', it is necessary and sufficient that condition (2) be satisfied.

Sufficiency is proved in the same way as in Theorem 1. In proving necessity, in addition to formula (6), the continuity of the arrangement of the points of the set $\overline{K}$ on the real plane and certain other properties of this set are used in an essential way.

Theorem 3. *Let Π' be the domain $0 \geq y \geq \varphi(x)$, where $\inf \varphi(x) > -\infty$. In order that the boundary-value problem (1) be stable in the domain Π', it is necessary and sufficient that the spectrum of the operator B lie in the left open half-plane.*

From the last two theorems the following easily follows. Let Π_2 be the domain $x \geq 0$, $y \leq 0$, $\inf \varphi(x) \geq -\infty$. Whatever the coefficients A, B, C may be, there exists at least one bounded right-hand side $f(x,y)$ in Π_2 and bounded initial data on $(0, \infty)$ such that the corresponding solution of the boundary-value problem (1) is unbounded in Π_2.

Using the methods developed in (^{3}), the results presented above can be extended to the case of equations with variable coefficients having weak variation at infinity (^{4}).

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[1] M. A. Rutman, DAN, 147, No. 4, 789 (1962).
[2] M. A. Rutman, Abstract of a doctoral dissertation, L., 1960, p. 7.
[3] M. A. Rutman, DAN, 108, No. 5, 770 (1956).
[4] K. P. Persidskii, Izv. Phys.-Math. Soc. at Kazan Univ., pt. 1, 8, ser. 3, 47 (1936).

Note: Figure translations are in progress. See original paper for figures.