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1965

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Abstract

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ON AN INTEGRAL EQUATION ON THE HALF-AXIS IN A CLASS OF FUNCTIONS GROWING AT INFINITY

(Presented by Academician N. I. Muskhelishvili, 29 VI 1964)

In the author' s papers (^{6a,b}) an infinite system of linear algebraic equations was considered in certain classes of growing sequences. The present work is devoted to transferring the results of the aforementioned papers to the continual analogue of this system—the integral equation of the form

$$\varphi(x) + \int_0^\infty [a(x-t) + b(x+t)]\varphi(t)\,dt = f(x) \qquad (0 \le x < \infty) \tag{1}$$

in the class of functions growing at infinity.

§ 1. Denote by $\mathcal{L}^{\langle\alpha\rangle}$ the normed ring (with convolution as multiplication) of complex-valued functions $a(x)$ such that

$$\|a\|_{L^{\langle\alpha\rangle}} = \int_{-\infty}^{\infty} |a(x)|\alpha(x)\,dx < \infty,$$

where $\alpha(x)$ is a continuous nondecreasing function satisfying the conditions

$$\alpha(x+t) \le \alpha(x)\alpha(t); \qquad \alpha(x) \ge 1; \qquad \lim_{|x|\to\infty} \frac{\ln\alpha(x)}{x} = 0. \tag{2}$$

In the ring $\mathcal{L}^{\langle\alpha\rangle}$ there are two subrings $\mathcal{L}_+^{\langle\alpha\rangle}$ and $\mathcal{L}_-^{\langle\alpha\rangle}$, consisting of those functions $a(x)$ from $\mathcal{L}^{\langle\alpha\rangle}$ for which $a(x) \equiv 0$ when $x < 0$, respectively $a(x) \equiv 0$ when $x > 0$.

Next, by $M^{\langle\alpha\rangle}$ denote the space conjugate to $\mathcal{L}^{\langle\alpha\rangle}$. It consists of measurable functions $f(x)$ for which almost everywhere on the axis the inequality $|f(x)| \le c_f\alpha(x)$ holds.

Analogously to $\mathcal{L}_\pm^{\langle\alpha\rangle}$, the spaces $M_\pm^{\langle\alpha\rangle}$ are introduced.

The ring of functions $a(x) \in \mathcal{L}^{\langle\alpha\rangle}$ is isomorphic (see (^{5})) to the ring $V_0^{\langle\alpha\rangle}$ of their Fourier transforms $A(x)^*$,

$$A(x) = \int_{-\infty}^{\infty} a(t)e^{itx}\,dt.$$

The ring obtained by extending $V_0^{\langle\alpha\rangle}$ through adjoining constants to it will be denoted by $V^{\langle\alpha\rangle}$. The rings $V_+^{\langle\alpha\rangle}$ and $V_-^{\langle\alpha\rangle}$ are introduced analogously.

We shall also consider the space $(V_0^{\langle\alpha\rangle})^*$, conjugate to $V_0^{\langle\alpha\rangle}$, the space of linear continuous functionals—generalized functions (g.f.) $F(x)$, defined on the space $V_0^{\langle\alpha\rangle}$. The g.f. $F(x)$ is defined as the generalized Fourier transform of a function $f(x) \in M^{\langle\alpha\rangle}$, given by the formula:

$$(F(x), A(x)) = 2\pi \int_{-\infty}^{\infty} f(t)a(-t)\,dt.$$

*

The properties of the function $A(x) \in V_0^{\langle\alpha\rangle}$ depend on the weight $\alpha(x)$; for example, when $\alpha(x) = (1+|x|)^s$, $s > 0$ an integer, $A(x)$ will be s times continuously differentiable, and $A(x)$ and its derivatives up to order s vanish at infinity. Examples of other $\alpha(x)$ satisfying conditions (2) are given in ([10]).

We shall call the generalized function $F^+(x)$ $(F^-(x))$ a generalized function of plus (minus) type ([3]), if it is the generalized Fourier transform of a function $\varphi(x) \in M_+^{\langle\alpha\rangle}$ $(M_-^{\langle\alpha\rangle})$.

It is not difficult to prove that a functional $F(x)$ is a generalized function of plus (minus) type if and only if it vanishes on all basic functions $A^+(x) \in V_{0+}^{\langle\alpha\rangle}$ $(A^-(x) \in V_{0-}^{\langle\alpha\rangle})$, i.e. $(F(x), A^+(x)) = 0$ $((F(x), A^-(x)) = 0)$.

§ 2. Consider the operators

$$A\varphi = \int_0^{\infty} a(x-t)\varphi(t)\,dt, \qquad B\varphi = \int_0^{\infty} b(x+t)\varphi(t)\,dt \quad (0 \le x < \infty).$$

For the operators A and B the following theorems hold:

Theorem 1. Suppose $a(x) \in \mathcal{L}^{\langle\alpha\rangle}$ and the condition

$$1 + A(x) = 1 + \int_{-\infty}^{\infty} a(t)e^{itx}\,dt \neq 0 \qquad (-\infty \le x \le \infty); \tag{3}$$

is satisfied. Then the operator $I + A$ (I is the identity operator) is a Φ-operator in the space $M_+^{\langle\alpha\rangle}$, and its index* $\varkappa(A)$ is computed by the formula

$$\varkappa(A) = -\operatorname{Ind}[1 + A(x)] = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\ln[1 + A(x)].$$

For the proof it suffices (see ([4])) to show that the Φ-operator is the operator A^*,

$$A^*\psi = \int_0^{\infty} a(t-x)\psi(t)\,dt,$$

in the space $\mathcal{L}_+^{\langle\alpha\rangle}$, for which the operator A is the adjoint. The proof of the latter fact rests on the Wiener–Levy theorem for the ring of functions $V^{\langle\alpha\rangle}$ (^{5}) and the solution of the problem of linear conjugation (the Riemann problem) in this ring (^{1,2,11,12}).

Theorem 2. Suppose $b(x) \in \mathcal{L}^{\langle\alpha\rangle}$; then the operator B is completely continuous in the spaces $\mathcal{L}_+^{\langle\alpha\rangle}$ and $M_+^{\langle\alpha\rangle}$.

We shall consider equation (1) under the assumption that $a(x), b(x) \in \mathcal{L}^{\langle\alpha\rangle}$, and $f(x)$ and $\varphi(x) \in M_+^{\langle\alpha\rangle}$. Using the operator notation, we rewrite (1) in the form $(I + A + B)\varphi = f$ $(\varphi, f \in M_+^{\langle\alpha\rangle})$.

Henceforth we shall everywhere assume that condition (3) is satisfied. Then from Theorems 1 and 2 it follows (see (^{4})) that $I + A + B$ is also a Φ-operator, and its index coincides with the index of the operator $I + A$.

Thus, we arrive at general theorems for equation (1).

Theorem 3. Let m be the number of solutions of the homogeneous equation (1), and m^* the number of solutions of the equation

$$\psi(x) + \int_0^\infty [a(t-x) + b(x+t)]\psi(t)\,dt = 0 \qquad (0 \le x < \infty), \tag{4}$$

for which equation (1) is the adjoint. Then the numbers m and m^* are finite, and for their difference the formula

$$m - m^* = \varkappa(A)$$

holds.

Theorem 4. For solvability of the nonhomogeneous equation (1) in the space $M_+^{\langle\alpha\rangle}$, it is necessary and sufficient that the conditions

$$\int_0^\infty f(x)\psi_j(x)\,dx = 0 \qquad (j = 1, 2, \dots, m^*), \tag{5}$$

be satisfied, where $\{\psi_j(x)\}$ is the totality of solutions from $\mathcal{L}_+^{\langle\alpha\rangle}$ of equation (4).

* The definition of a Φ-operator and its index is given in (^{4}).

From Theorems 3-4 and property (2) of the weight $\alpha(x)$ it follows:

Theorem 5. *The homogeneous equation (1) in the class* $M_+^{\langle\alpha\rangle}$ *has no solutions other than solutions from the ring* $\mathcal{L}^{\langle\alpha\rangle}$, *to which the kernels* $a(x)$ *and* $b(x)$ *belong.*

§ 3. Passing in equation (1) to generalized Fourier transforms, we arrive at the problem:

Find a generalized function of plus type $\Phi^+(x)$ and a generalized function of minus type $\Phi^-(x)$ from the relation*:

$$\Phi^+(x)[1+A(x)]+\Phi^+(-x)B(x)=\Phi^-(x)+F(x), \tag{6}$$

where $A(x), B(x) \in V_0^{\langle\alpha\rangle}$, and $F(x) \in (V_0^{\langle\alpha\rangle})^*$.

Equation (4) is equivalent to the boundary-value problem:

$$\Psi^+(x)[1+A(-x)]+\Psi^+(-x)B(x)=\Psi^-(x) \quad (-\infty<x<\infty), \tag{7}$$

where $\Psi^\pm(x) \in V_{0\pm}^{\langle\alpha\rangle}$.

Problems (6) and (7) reduce to the Carleman problem for a pair of functions and therefore, generally speaking, are not solved in closed form (^{6a,b,7}). Therefore the further study of equation (1) is carried out under additional conditions on $a(x)$ and $b(x)$.

§ 4, 1. Let $a(x)$ and $b(x)$ satisfy the conditions:

$$A(-x)=\overline{A(x)}, \qquad |1+A(x)|>|B(x)| \quad (-\infty\le x\le\infty); \tag{8}$$

then the following holds:

Theorem 6. *If $\varkappa(A)>0$, then the homogeneous equation (1) has $m=\varkappa(A)>0$ solutions, and the nonhomogeneous equation is unconditionally solvable ($m^*=0$). If $\varkappa(A)\le 0$, then $m=0$, and the nonhomogeneous equation (1) is uniquely solvable only when $m^*=|\varkappa(A)|$ conditions (5) are fulfilled.*

The proof of Theorem 6 is based on Lemmas 1 and 2.

Lemma 1. *Suppose the first condition (8) is fulfilled. Then the function $1+A(-x)$ is representable in the form $1+A(-x)=X^-(x)[X^+(x)]^{-1}$, $X^\pm(x)\in V_\pm^{\langle\alpha\rangle}$, where $X(z)$ is the canonical function of the homogeneous Riemann problem*

$$\omega^+(x)[1+A(-x)]=\omega^-(x) \quad (-\infty<x<\infty), \tag{9}$$

and

$$|X^+(-x)|=|X^+(x)| \quad (-\infty\le x\le\infty).$$

Lemma 2. *Suppose condition (8) is fulfilled. Then for $\varkappa(A)\ge 0$ the problem (7) has no nontrivial solutions in the class $\mathcal{L}^2(-\infty,\infty)$.*

Proof**. Assuming that in the boundary condition (7) $\Psi^\pm(x)\in\mathcal{L}^2_\pm$, by the substitution

$$\Psi(z)X^{-1}(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{g(t)}{t-z}\,dt,$$

where $X(z)$ is the canonical function of the Riemann problem (9), and $g(t)\in\mathcal{L}^2$, the boundary condition (7) can be given the form of the equation

$$g(x)=-\frac{B(x)}{1+A(-x)}\frac{X^+(-x)}{X^+(x)}\left[\frac{1}{2}g(-x)+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{g(t)}{t+x}\,dt\right], \tag{10}$$

which may be regarded as an operator equation.

Applying Lemma 1, condition (8), and observing that the norm of the singular operator S,

$$Sg = \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{g(t)\,dt}{t-x},$$

is equal to one in the space $\mathcal{L}^2$, we obtain that the operator on the right-hand side of (10) will be a contraction operator. Therefore, by the contraction mapping principle, equation (10) (and hence problem (7)) has no nontrivial solutions in $\mathcal{L}^2$.

* The product of a generalized function $F(x)$ by the base $B(x)$ is defined by the rule

$$(F(x)B(x), A(x)) = (F(x), B(x)A(x)).$$

** The proof of Lemma 2 is analogous to the method of B. V. Boyarskii (^{8}).

The proof of Theorem 6 is now obtained if one takes into account that, in the classes $\mathcal{L}^{+2}$ and $\mathcal{L}_+^{\langle\alpha\rangle}$, the homogeneous equations (1) and (4), by Theorem 5, have the same solutions, and that the index of the coefficient at $\Phi^+(x)$ in problem (6) differs in sign from the index of the Riemann problem (9).

2. If the Fourier transforms of the kernels $a(x)$ and $b(x)$ are connected by the dependence

$$[1 + A(x)][1 + A(-x)] = [1 + B(x)][1 + B(-x)] \neq 0 \tag{11}$$

$$(-\infty \leq x \leq \infty),$$

then the solution of equation (1) is found effectively; moreover, the following holds.

Theorem 7. Denote

$$\varkappa_\pm = \operatorname{Ind} G_\pm(x) = \pm \operatorname{Ind}[1 + B(x)] - \operatorname{Ind}[1 + A(x)],$$

$\varkappa'_\pm$ is the number $\frac{1}{2}\varkappa_\pm$, if $\varkappa_\pm$ is even, and $\frac{1}{2}(\varkappa_\pm + 1)$, if $\varkappa_\pm$ is odd. Then: 1) if $\varkappa_\pm \geq 0$, then $m = \varkappa(A) > 0,\ m^* = 0$; 2) if $\varkappa_\pm < 0$, then $m = 0,\ m^* = |\varkappa(A)|$; 3) if $\varkappa_- < 0,\ \varkappa_+ > 0$, then $m = \varkappa'_+$ for even $\varkappa_+$ and $m = \varkappa'_+ - 1$ for odd $\varkappa_+$, while m^* is respectively equal to $|\varkappa_-|$ and $|\varkappa'_-| + 1$; 4) if $\varkappa_- > 0,\ \varkappa_+ < 0$, then $m = \varkappa'_- - r$ ($m = \varkappa'_- - 1 - \tilde{r}$ for odd $\varkappa_-$), and $m^* = -\varkappa'_+ - r$ ($m^* = -\varkappa'_+ - 1 - \tilde{r}$), where r ($\tilde{r}$) is the rank of a matrix whose elements are explicitly expressed in terms of $A(x)$ and $B(x)$.

Let us note that the proof of Theorem 7 is based on the properties of the Carleman problems in the class o.f. $(V_0^{\langle\alpha\rangle})^*$:

$$\Phi^{-}(-x) = G_{-}(x)\Phi^{-}(x) + G_{-}(x)F(x) - F(-x),$$

$$\Phi^{+}(x) = -G_{+}(x)\Phi^{+}(-x) + [F(x) + \Phi^{-}(x)][1 + A(x)]^{-1},$$

to which problem (6) is equivalent under condition (11) (see (6a, c)).

3. We shall say that a function $A(x) \in V^{\langle\alpha\rangle}$ possesses property Γ_{+} (see [9]) if it is representable in the form $P(x)/Q(x)$, where $P(x), Q(x) \in V_{+}^{\langle\alpha\rangle}$ and are nonzero for $-\infty \leq x \leq \infty$. Let us note that all rational functions from V possess property Γ_{+}.

An effective solution of problem (6) (equation (1)) is possible under the following conditions*:

1°. When the kernel $b(x)$ of equation (1), given only for $x \geq 0$, is continued by zero to $x < 0$, the function

$$T(x) = [1 + A(x)][1 + A(-x)] - [a + B(x)][a + B(-x)]$$

($a \neq 1$ is an arbitrary number) possesses property Γ_{+}.

2°. For some continuation of $b(x)$, the function $a + B(x) \neq 0$ for $-\infty \leq x \leq \infty$, and the function

$$T(x)[a + B(x)]^{-1}$$

possesses property Γ_{+}.

Let us note that in cases 1°, 2° problem (6) is equivalent to a Riemann problem in the class o.f. $(V^{\langle\alpha\rangle})^{*}$ and to some additional conditions. However, in contrast to § 4, 2, an exact count of the number of solutions and of the solvability conditions of equation (1), as in Theorem 7, is impossible.

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Received
1 VI 1964

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* Under analogous conditions, equation (1) was investigated in the class $L(-\infty, \infty)$ by I. A. Feldman (9).

Note: Figure translations are in progress. See original paper for figures.

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