ON THE CONTINUAL PRODUCT OF LEBESGUE SPACES

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Abstract

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ON THE CONTINUAL PRODUCT OF LEBESGUE SPACES

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Denote by τ the cardinality of the continuum, and by E^τ the direct product of a continuum of unit intervals with the usual Lebesgue measure on them. In this note we consider conditions under which a space with measure is isomorphic to E^τ.

Let E be a space with measure m, defined on a σ-algebra of measurable sets, such that the following conditions are satisfied:

1) $mE = 1$;

2) if $A \subset B$, where $B \in Q$ and $mB = 0$, then $A \in Q$.

The **metric weight** of the space E is the least of the cardinal numbers λ for which there exists a system $\mathfrak{B}$ of cardinality λ of measurable sets such that, for every measurable set A, there is a countable sequence $A_1, A_2, \dots, A_n, \dots$ of sets from $\mathfrak{B}$ for which

$$m((A \cap \overline{A}_n) \cup (\overline{A} \cap A_n)) \to 0.$$

The space E is called **homogeneous** if every measurable subset of it of positive measure, considered as a space with measure, has metric weight equal to the metric weight of the space E.

The **exact weight** of the space E is the least of the cardinal numbers λ for which there exists a system Σ of cardinality λ of measurable sets such that:

a) every measurable set A has the form

$$A = A^1 \cup A^2,$$

where A^1 belongs to the smallest σ-algebra H containing Σ, and $A^2 \subset B$, where $B \in H$ and $mB = 0$;

b) for any two points of E there is a set in Σ containing one point and not containing the other. A system Σ having properties a) and b) and having cardinality equal to the exact weight of the space will be called a **basis** of E.

A space isomorphic to E^τ must satisfy, in addition to 1) and 2), the following conditions:

3) the space E is homogeneous;

4) the metric weight of E is equal to the exact weight of E and is equal to τ.

Finally, the last condition is as follows:

5) there exists a class $\Phi = \{F\}$ of subsets of the space E satisfying the requirements:

a) every centered countable sequence of sets from Φ has a nonempty intersection;

b) the intersection of a countable number of sets from Φ, if it is nonempty, belongs to Φ;

c) each set from Φ has cardinality 2^τ;

d) for every measurable set A of positive measure, set

$$mA = \sup_{F \in \Phi,\, F \subset A} m_{xF},$$

where

$$m_{xF} = \sup_{F \in Q,\, B \subset F} mB.$$

There exists an example of a space with measure that is not isomorphic to the space E^t, but satisfies conditions 1)–4) and the compactness condition of the measure in the sense of Marczewski ([1]), which consists in the fact that in E there exists a class Φ satisfying requirements a), b), and d) of condition 5.

This example can also be made so that the class Φ will satisfy requirements a), c), and d), but will not satisfy requirement b).

Theorem. A space with measure satisfying conditions 1)–5) is isomorphic to the space E^t.

Proof. For a system R of sets, denote by $\zeta(R)$ the partition of the space E, every element of which either is contained in, or has nonempty intersection with, every set from R, and for any two elements of $\zeta(R)$ there is a set from R containing one element and not containing the other.

We shall call a σ-algebra $R \subset Q$ **Lebesgue** if the quotient space with respect to the partition $\zeta(R)$ is isomorphic to the unit interval with Lebesgue measure.

Let R be a Lebesgue σ-algebra, Σ'_R a basis of the quotient space with respect to the partition $\zeta(R)$, and Σ_R the system of sets that are inverse images of sets from Σ'_R under the mapping $T_{\zeta(R)}$ (T_ζ is the mapping of E onto E/ζ, sending each point of E to that element of the partition ζ in which it is contained).

Construct a countable algebra $\mathfrak{B}$ of measurable sets such that:

a) $\Sigma_R \subset \mathfrak{B}$;

b) for every set $A \in \mathfrak{B}$ of positive measure and every $\varepsilon > 0$, there is a $B \in \mathfrak{B}$ such that $B \subset A$, $m(A \setminus B) < \varepsilon$, and there exists a set $F \in \Phi$ for which $B \subset F \subset A$.

Denote by $S(R)$ the smallest σ-algebra containing $\mathfrak{B}$. It is easy to see that almost all elements of the partition $\zeta(S(R))$ belong to Φ.

Take some basis Σ of the space E and arrange the sets from Σ into a transfinite sequence

$$Z_1, \dots, Z_\lambda, \dots, \quad \lambda < \omega_\tau.$$

Now take some Lebesgue σ-algebra R' containing Z_1, and construct $S_1 = S(R)$. Put $S' = S_1$, and for all $\lambda < \omega_\tau$ construct Lebesgue σ-algebras S^λ such that:

a) if $\lambda' < \lambda''$, then $S^{\lambda'} \subset S^{\lambda''}$;

b) $Z_\lambda \in S^\lambda$;

c) almost all elements of the partition $\zeta(S^\lambda)$ belong to Φ;

d) every element of the partition

$$\prod_{\gamma<\lambda} \zeta(S^\gamma),$$

belonging to Φ, contains a continuum of elements of the partition $\zeta(S^\lambda)$;

e) there exists a null set $M_\lambda \in S^\lambda$ which intersects each element of the partition

$$\prod_{\gamma<\lambda} \zeta(S^\gamma),$$

having cardinality 2^τ, in a set of cardinality 2^τ.

We shall now use the following proposition, which is a slight generalization of one lemma from ([2]). Let ζ_1 and ζ_2 be two measurable partitions of a Lebesgue space, whose product is the partition into points. Then there exists a measurable partition $\zeta'_2 = \zeta_2 \pmod 0$ such that the product $\zeta_1 \times \zeta'_2$ is the partition into points, and every element of ζ_1 having a continuum of points has a nonempty intersection with every element of ζ_2'.

With the aid of this proposition, for each λ we construct a Lebesgue σ-algebra $S_\lambda \subset S^\lambda$ such that every element of the partition belonging to Φ

$$\prod_{\gamma<\lambda} \zeta(S^\gamma)$$

has nonempty intersection with each element of $\zeta(S_\lambda)$.

Let us now take the space $\widetilde{E}$, whose points are all possible sequences of elements of the partitions $\zeta(S_\lambda)$,

$$(c^\lambda) = c^1, \dots, c^\lambda, \dots, \quad \lambda < \omega_\tau,$$

one taken from each partition. Having defined, for arbitrary sets,

$$\widetilde{A}_1 = \{(c^\lambda) : c^{\lambda_1} \subset A^1 \in S_{\lambda_1}\}, \dots, \quad \widetilde{A}_n = \{(c^\lambda) : c^{\lambda_n} \subset A^n \in S_{\lambda_n}\},$$

$$m\left(\bigcap_{i=1}^n \widetilde{A}_i\right) = m\left(\bigcap_{i=1}^n A^i\right)$$

and continuing in the usual way, we define a measure in $\widetilde{E}$. In (2) it is shown that the measure space so defined is isomorphic to the space E^τ. Therefore it remains only to prove that the space E is isomorphic to the space $\widetilde{E}$. We shall construct an isomorphic mapping of E onto $\widetilde{E}$ as follows. Let M^λ be the sum of all elements of the partition $\zeta(S^\lambda)$ not belonging to Φ. Let c_0 be some element of the partition $\zeta(S_1)$ having cardinality 2^τ. We define

$$M_{(1)} = M^1 \cup c_1 \cup c_0, \quad M_{(\lambda)} = M^\lambda \cup M_\lambda \cup c_\lambda, \quad N_\lambda = M_{(\lambda)} \setminus \left(\bigcup_{\gamma<\lambda} M_{(\gamma)}\right).$$

Now define

$$\widetilde{N}_\lambda = \{(c^\alpha) : \bigcap_{\alpha<\lambda} c^\alpha \in N_\lambda\}$$

and denote by U_λ a one-to-one mapping of N_λ onto $\widetilde{N}_\lambda$, under which, for any element c of the partition

$$\prod_{\gamma<\lambda} \zeta(S^\gamma),$$

having nonempty intersection with N_λ,

$$U_\lambda(C \cap N_\lambda) = \{(c^\alpha) : \bigcap_{\alpha<\lambda} c^\alpha = c\} \cap \widetilde{N}_\lambda.$$

The mapping of the space E onto $\widetilde{E}$ which coincides on each N_λ with U_λ is an isomorphic mapping of E onto $\widetilde{E}$.

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CITED LITERATURE

1. E. Marczewski, *Fund. Math.*, **40**, 113 (1953).

2. V. G. Vinokurov, DAN, **158**, No. 6 (1964).

Note: Figure translations are in progress. See original paper for figures.