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Abstract

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DETERMINATION OF A STURM-LIOUVILLE EQUATION WITH A SINGULARITY FROM TWO SPECTRA

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1. V. A. Marchenko proved ([1]) that two different spectra of one singular Sturm–Liouville equation uniquely determine this equation. Later the author and B. M. Levitan, in the paper ([2]), indicated an effective method for constructing a singular equation from two spectra. However, these constructions are conditional in character, since it is assumed in advance that two sequences of numbers $\lambda_0 < \lambda_1 < \cdots$ and $\mu_0 < \mu_1 < \cdots$ are eigenvalues of one and the same equation. Therefore it is of interest to find conditions on the numbers $\{\lambda_n\}$ and $\{\mu_n\}$ under which they are two different spectra of one equation of a definite type. In the case of a regular equation this problem has been completely solved in papers ([3–7]), and in the case of a singular equation defined on a half-axis, partially in paper ([7]). Paper ([8]) contains a survey of these results.

In the present paper we give a complete solution of the problem posed above in the case when the Sturm–Liouville equation is defined on the finite interval $[0, \pi]$, but at the point π has a singularity of the type $l(l+1)/(\pi-x)^2$, where l is a positive integer. We note that in this paper we first solve the ordinary inverse Sturm–Liouville problem for such equations. The ordinary inverse problem for an equation with a singularity of the indicated type at both endpoints of the interval $(0, \pi)$ was solved in papers ([10,11]).

2. Consider the differential equation

$$-y'' + \{l(l+1)/(\pi - x^2 + q(x)\}y = sy \tag{1}$$

and the boundary conditions

$$y'(0) - hy(0) = 0; \tag{2}$$

$$y(\pi) = 0. \tag{3}$$

Here it is assumed that $q(x)$ is a real function summable with its square on the interval $[0, \pi]$; l is a positive integer; s is a complex parameter; h is a real number. We shall denote the eigenvalues of problem (1)–(2)–(3) by $\lambda_0 < \lambda_1 < \cdots$, and the corresponding eigenfunctions normalized by the condition $y(0) = 1$, by $\varphi_n(x)$. The numbers

$$\alpha_n = \int_0^\pi \varphi_n^2(x)\,dx \tag{4}$$

are called the norming constants of problem (1)–(2)–(3).

Theorem 1. In order that the sequences of numbers $\lambda_0 < \lambda_1 < \cdots$ and $a_0, a_1, \ldots$ respectively be the eigenvalues and norming constants of a boundary-value problem of the type (1)–(2)–(3) with a function $q(x) \in L_2(0, \pi)$, it is necessary and sufficient that the following conditions be fulfilled:

1°. The numbers $a_0, a_1, \ldots$ are positive.

2°. For large n the asymptotic formulas hold

$$\lambda_n = (n + l/2)^2 + a + a_n; \tag{5}$$

$$\alpha_n = \pi/2 + b_n/n, \tag{6}$$

where a is a constant number, and the series

$$\sum_{n=0}^{\infty} a_n^2, \qquad \sum_{n=0}^{\infty} b_n^2 \tag{7}$$

converge.

The necessity of the conditions of the theorem follows from the known asymptotics for λ_n and α_n. The sufficiency follows from the subsequent theorems.

Theorem 2. If the sequences of numbers $\lambda_0, \lambda_1, \ldots$ and $\alpha_0, \alpha_1, \ldots$ are respectively the eigenvalues and norming constants of the boundary-value problem of type (1)–(2)–(3) for $l = 0$ with function $q_1(x) \in L_2(0, \pi)$, then the numbers $\lambda_p, \lambda_{p+1}, \ldots$ and $\alpha_p, \alpha_{p+1}, \ldots$ are respectively the eigenvalues and norming constants of the problem of type (1)–(2)–(3) with potential

$$2p(2p + 1)/(\pi - x)^2 + q(x), \tag{8}$$

where $q(x) \in L_2(0, \pi)$, and conversely, if the numbers $\lambda_p, \lambda_{p+1}, \ldots$ and $\alpha_p, \alpha_{p+1}, \ldots$ are the eigenvalues and norming constants of the problem of type (1)–(2)–(3) for $l = 2p$ and $q(x) \in L_2(0, \pi)$, then the numbers $\lambda_0, \lambda_1, \ldots, \lambda_p, \ldots$

and $\alpha_0, \alpha_1, \dots, \alpha_p, \dots$, where $\lambda_0, \lambda_1, \dots, \lambda_{p-1}$ are arbitrary distinct numbers different from $\lambda_p, \lambda_{p+1}, \dots$, and $\alpha_0, \alpha_1, \dots, \alpha_{p-1}$ are arbitrary positive numbers, are the eigenvalues and norming constants of the boundary-value problem of type (1)–(2)–(3) for $l = 0$.

Theorem 3. If the sequences of numbers $\lambda_0, \lambda_1, \dots$ and $\alpha_0, \alpha_1, \dots$ are respectively the eigenvalues and norming constants of the boundary-value problem

$$-y'' + q_1(x)y = sy; \tag{9}$$

$$y'(0) - hy(0) = 0; \tag{10}$$

$$y'(\pi) + Hy(\pi) = 0 \tag{11}$$

without singularities, then the numbers $\lambda_p, \lambda_{p+1}, \dots$ and $\alpha_p, \alpha_{p+1}, \dots$ are the eigenvalues and norming constants of the problem of type (1)–(2)–(3) for $l = 2p - 1$; conversely, if the numbers $\lambda_p, \lambda_{p+1}, \dots$ and $\alpha_p, \alpha_{p+1}, \dots$ are the eigenvalues and norming constants of the boundary-value problem (1)–(2)–(3) for $l = 2p - 1$, then the numbers $\lambda_0, \lambda_1, \dots, \lambda_p, \dots$ and $\alpha_0, \alpha_1, \dots, \alpha_p, \dots$, where $\lambda_0, \lambda_1, \dots, \lambda_{p-1}$ are distinct arbitrary numbers different from the numbers $\lambda_p, \lambda_{p+1}, \dots$, and $\alpha_0, \alpha_1, \dots, \alpha_{p-1}$ are arbitrary positive numbers, are the eigenvalues and norming constants of the boundary-value problem of type (9)–(10)–(11) without singularities.

The converse assertions of these theorems follow from the asymptotic formulas (5), (6) and from the results of [9] (see also [8]).

We now outline the proofs of the direct assertion, for example of Theorem 2. Let $\lambda_p, \lambda_{p+1}, \dots$ and $\alpha_p, \alpha_{p+1}, \dots$ be the eigenvalues and norming constants of the boundary-value problem (1)–(2)–(3) for $l = 2p$, and let $\varphi_p(x), \varphi_{p+1}(x), \dots$ be the eigenfunctions normalized by the condition $\varphi_n(0) = 1$. Then one can verify that the functions

$$\psi_n(x) = \varphi_n(x) + \frac{\varphi_p(x)}{\alpha_p + \int_0^x \varphi_p^2(t)\,dt} \int_0^x \varphi_p(y)\varphi_n(y)\,dy \tag{12}$$

$$(n = p + 1,\ p + 2, \dots)$$

are eigenfunctions of an operator of type (1)–(2)–(3) for $l = 2p + 2$, and prove our assertions.

We note that the transformation (12) makes it possible to reduce the study of the inverse problem for equation (1) with a singularity at the point π to the study of the inverse problem for an equation without singularities.

3. In this section results are formulated that give a complete solution of the inverse problem from two spectra for equation (1). In the boundary condition (2) we replace the number h by the number h_1. We obtain a new boundary-value problem, whose eigenvalues we denote by $\mu_0, \mu_1, \ldots$. By the methods of papers ([6–8]) and using the results of Sec. 2, one can prove the following theorem.

Theorem 4. In order that the sequences of numbers $\lambda_0, \lambda_1, \ldots$ and $\mu_0, \mu_1, \ldots$ be eigenvalues of one and the same equation of type (1) with an integer l and with potential $q(x) \in L_2(0,\pi)$, the following conditions are necessary and sufficient:

1°. The numbers λ_n and μ_n interlace.

2°. The asymptotic formulas

$$\lambda_n = \left(n + \frac{l}{2}\right)^2 + a + a_n, \qquad \mu_n = \left(n + \frac{l}{2}\right)^2 + b + b_n,$$

hold, where $a \neq b$, and the series $\sum_{n=0}^{\infty} a_n^2$ and $\sum_{n=0}^{\infty} b_n^2$ converge.

3°. The difference

$$\mu_n - \lambda_n = b - a + \frac{c_n}{n},$$

where the series $\sum_{n=1}^{\infty} c_n^2$ converges.

Furthermore, if $q(x)$ has m summable derivatives, then the function

$$F(x) = \sum_{n=0}^{\infty} \left\{ \frac{\mu_n - \lambda_n}{b-a} \cos \sqrt{\lambda_n}\, x - \cos\left(n + \frac{l}{2}\right) x \right\}$$

has m absolutely continuous derivatives on the interval $(0, 2\pi)$, and, conversely, if $F(x)$ has m absolutely continuous derivatives on the interval $[0, 2\pi]$, then $q(x)$ has m summable derivatives on the interval $[0, \pi]$.

Corollary 1. If the numbers λ_n and μ_n interlace and

$$\lambda_n = \left(n + \frac{l}{2}\right)^2 + a + \frac{a_1}{n^2} + O(1/n^3),$$

$$\mu_n = \left(n + \frac{l}{2}\right)^2 + b + \frac{b_1}{n^2} + O(1/n^3),$$

where $a \neq b$, a_1, b_1 are constant numbers, then they are eigenvalues of one and the same equation of type (1) with different boundary conditions at zero and with an absolutely continuous function $q(x)$.

Apparently, Theorem 4 and its corollary remain valid also in the case of nonintegral $l \geq \frac{1}{2}$. But we have not succeeded in proving this hypothesis. We have only succeeded in proving the following theorem.

Theorem 5. If the numbers λ_n and μ_n interlace and

$$\lambda_n = \left(n + \frac{l}{2}\right)^2 + a + \frac{a_1}{n^2} + O(1/n^3),$$

$$\mu_n = \left(n + \frac{l}{2}\right)^2 + b + \frac{b_1}{n^2} + O(1/n^3),$$

where l is any positive number greater than $1/2$, $a \neq b$, and a_1, b_1 are constant numbers, then the numbers λ_n and μ_n are eigenvalues of one and the same equation of the form

$$-y'' + q(x)y = sy \qquad (0 \leq x < \pi),$$

where, generally speaking, the function $q(x)$ has a singularity at the point π.

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Note: Figure translations are in progress. See original paper for figures.