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Abstract

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THEORY OF ELASTICITY

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ON THE PROPERTIES OF THE GENERAL EQUATIONS OF THE THEORY OF IDEAL PLASTICITY

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The paper considers the conditions for the possible existence of surfaces of weak discontinuity for the general case of the stressed and deformed state of an ideal isotropic incompressible rigid-plastic body under arbitrary smooth and piecewise-smooth yield conditions and the associated flow law. On surfaces of weak discontinuity, by assumption, the derivatives of the components of the stresses and strain rates undergo a jump. It is shown that, for arbitrary sufficiently smooth yield surfaces, surfaces of weak discontinuity can exist only in cases when the stressed state at a point is simple shear. This result for the Mises yield condition was obtained earlier in work ([1]). For the case of an arbitrary edge formed by the intersection of smooth yield surfaces in the space of principal stresses, the corresponding analysis coincides with the case of edges of piecewise-smooth yield surfaces ([2]). The case of an axisymmetric stressed state is also considered for arbitrary smooth and piecewise-smooth yield surfaces that depend also on the first invariant of the stress tensor.

1. Consider the equations of the theory of an isotropic ideal incompressible rigid-plastic body. The equilibrium equations have the form

$$\sigma_{ij,j} + F_i = 0. \tag{1,1}$$

We shall write the yield condition in the form

$$f^{(p)}(\sigma_{ij}) = 0, \qquad p = 1, 2, \dots, k. \tag{1,2}$$

The relations between stresses and strain rates are determined from the associated plastic-flow law

$$\varepsilon_{ij} = \lambda_p \partial f^{(p)}/\partial \sigma_{ij}, \qquad \lambda_p \geq 0, \qquad \varepsilon_{ii} = 0, \qquad \varepsilon_{ij} = {}^1/_2(u_{i,j} + u_{j,i}). \quad (1,3)$$

We shall assume that the field of stresses and strain rates is continuous. Consider characteristic surfaces of weak discontinuity $\chi(x_i) = 0$, on which the first derivatives of the stresses and strain rates undergo discontinuity. Suppose that the yield surface is a smooth, twice-differentiable function of its arguments. In what follows we shall use the expressions of the yield condition and of the associated flow law in components of the principal stresses and strain rates, which we shall denote respectively by σ_i, ε_i. We shall write the yield condition in the form

$$f(\sigma_i) = 0. \quad (1,4)$$

The yield condition (1,4) is interpreted in the space of principal stresses as a cylindrical surface whose generators are parallel to the line $\sigma_1 = \sigma_2 = \sigma_3$, equally inclined to the principal axes.

For the components of the stresses and strain rates we shall have

$$\sigma_{ij} = c_{ik} c_{jk} \sigma_t, \qquad \varepsilon_{ij} = c_{ik} c_{jk} \varepsilon_t \quad (t = k), \quad (1,5)$$

where c_{ij} are the direction cosines of the principal directions in the coordinate system x_i. The quantities c_{ij} satisfy the orthogonality relations

$$c_{ik} c_{jk} = \delta_{ij} \quad \text{or} \quad c_{ij} c_{ik} = \delta_{jk}. \quad (1,6)$$

The relations of the associated flow law take the form

$$\varepsilon_{ij} = \lambda c_{ik} c_{jk} \partial f/\partial \sigma_t \quad (t = k). \quad (1,7)$$

We note that in what follows the yield function is differentiated only with respect to the stress components; we shall denote

$$\partial f/\partial \sigma_i = f_i, \qquad \partial f/\partial \sigma_i \partial \sigma_j = f_{ij}, \qquad f_1 + f_2 + f_3 = 0. \quad (1,8)$$

We obtain the initial system of equations in the following way. Substituting relations (1,5) into the equilibrium equations (1,1), we obtain

$$(c_{ik} c_{jk} \sigma_t)_{,j} + F_i = 0 \quad (k = t). \quad (1,9)$$

Differentiating the relations of the associated flow law (1,7) in any fixed direction n, we find

$$u_{i,jn} + u_{j,in} = 2(\lambda c_{ik} c_{jk} f_t)_{,n} \tag{1,10}$$

where, for the derivatives of the direction cosines, according to (1,6), one has

$$(c_{ik} c_{jk})_{,n} = 0. \tag{1,11}$$

Differentiating the yield condition (1,4), we obtain

$$f_i \sigma_{i,n} = 0. \tag{1,12}$$

The system of 16 equations (1,9)–(1,12) in the 16 unknowns σ_i, u_i, c_{ij}, λ is closed. The kinematic compatibility conditions on a surface of weak discontinuity have the form ([1])

$$[\sigma_{i,j}] = \xi_i \alpha_j, \qquad [u_{i,jn}] = \eta_i \alpha_j \alpha_n, \qquad [c_{ij,n}] = \zeta_{ij} \alpha_n, \tag{1,13}$$

where ξ_i, η_i, ζ_{ij} are the components of the characteristic segments determining the magnitudes of the discontinuities; α_i are the direction cosines of the normal to the surface of weak discontinuity.

Writing equations (1,9)–(1,12) in the discontinuities, we obtain a system of homogeneous equations in the unknowns ξ_i, η_i, ζ_{ij}. Expanding the characteristic determinant, we obtain an equation for the direction cosines of the normal α_i to the surface of weak discontinuity. Substantial simplifications are achieved when the canonical coordinate system is used; in this case the axes x_i coincide with the principal axes of the stress and strain-rate tensors. The required equation in the canonical coordinate system will have the form

$$\begin{gathered}\alpha_1^2 \beta_2 \beta_3 (f_3 \alpha_2^2 + f_2 \alpha_3^2)^2 + \alpha_2^2 \beta_3 \beta_1 (f_1 \alpha_3^2 + f_3 \alpha_1^2)^2 + \alpha_3^2 \beta_1 \beta_2 (f_2 \alpha_1^2 + f_1 \alpha_2^2)^2 \\ -\Phi\,[\beta_1 \alpha_1^2 (1 - 2\alpha_1^2)^2 + \beta_2 \alpha_2^2 (1 - 2\alpha_2^2)^2 + \beta_3 \alpha_3^2 (1 - 2\alpha_3^2)^2] = 0,\end{gathered} \tag{1,14}$$

$$\Phi = f_{12} f_3^2 + f_{23} f_1^2 + f_{31} f_2^2, \qquad \beta_i = (f_j - f_k)/s_i, \qquad s_i = \sigma_j - \sigma_k.$$

Restricting ourselves to the class of uncurved yield surfaces, and taking account of the incompressibility condition, it is easy to show that at points of the surface $f(\sigma_i) = 0$ at which one of the principal curvatures is positive (the other being zero), $\Phi < 0$. In the case where both principal curvatures are zero (points of straightening), $\Phi = 0$. For isotropic bodies, by analogy with ([2]), it follows that, generally speaking, $\beta_i > 0$. In this case equation (1,14) can be satisfied only in the following cases:

$$\alpha_i = 0, \qquad \alpha_j = \sqrt{2}/2, \qquad f_i = 0. \tag{1,15}$$

From (1,15) and the incompressibility condition it follows that $f_j + f_k = 0$.

Thus, the original equations have characteristic directions at a given point if, in its neighborhood, the yield condition has the form $f(\sigma_j - \sigma_k) = 0$.

Consequently, surfaces of weak discontinuity, for arbitrary smooth yield surfaces for an ideal isotropic incompressible rigid-plastic body, exist only in the case when the stressed state at the point corresponds to simple shear. In this case the characteristic surfaces coincide with the planes of maximum shear stresses $\tau_{\max} = 1/2(\sigma_j - \sigma_k)$. This result for the case of the Mises plasticity condition was obtained in (1).

In the case when $\beta_i = 0$, $\beta_i > 0$, $\beta_k > 0$, equation (1.14) has the additional root $\alpha_i = 1$, $\alpha_j = \alpha_k = 0$, whence it follows that the planes of the principal stresses are characteristic; in the neighborhood of such points the plasticity condition must have the form $f(2\sigma_i - \sigma_j - \sigma_k) = 0$.

In the case of faces of piecewise-linear yield conditions $a_i\sigma_i = \text{const}$, $a_i = \text{const}$, the function $\Phi = 0$. This case was considered in (2).

Let the stressed and deformed state correspond to an edge of an arbitrary piecewise-smooth yield surface in the space of principal stresses. In this case the yield conditions are

$$f(\sigma_i) = 0, \qquad g(\sigma_i) = 0. \tag{1.16}$$

The relations of the associated law of plastic flow have the form

$$\varepsilon_{ij} = \lambda c_{ik} c_{jk} t + \mu c_{ik} c_{jk} g_t, \qquad \lambda > 0, \quad \mu > 0 \quad (t = k). \tag{1.17}$$

Similarly, differentiating relations (1.16), (1.17) in an arbitrary direction n, adjoining relations (1.9), (1.11), we obtain the original system of equations. Passing to discontinuities of the derived components of stresses and strain rates, expanding the characteristic determinant, in the canonical coordinate system we shall have

$$A_1\alpha_1^2(1-2\alpha_1^2)^2 + A_2\alpha_2^2(1-2\alpha_2^2)^2 + A_3\alpha_3^2(1-2\alpha_3^2)^2 = 0, \tag{1.18}$$
$$A_i = (\varepsilon_j - \varepsilon_k)s_{js}k \qquad (\text{no summation over } j, k).$$

Only the first derivatives of the yield functions enter equations (1.18). Consequently, the analysis of the characteristic equation for the edges of piecewise-linear yield conditions, carried out in (2), remains fully valid for the case of an edge formed by piecewise-smooth yield surfaces.

The case when the stressed and deformed state corresponds to three yield conditions $f^{(p)}(\sigma_{ij})$, $p = 1, 2, 3$, belongs to the class of statically determinate problems.

The sole possibility of such a state was considered in (3). Indeed, the system of 3 equilibrium equations with respect to the stress components will be closed if all stress components can be expressed through no more than 3 independent variables. To the stress tensor in stress space there may be put in correspondence a stress ellipsoid. If the stress ellipsoid is an ellipsoid of revolution, and its third semiaxis is determined as a function of the other axes, then the stress ellipsoid is completely determined by the magnitude of one of the axes and by the orientation of the third semiaxis, determined by 2 Euler angles. This case corresponds to the yield condition $\sigma_i = \sigma_j$, $\sigma_k = f(\sigma_i)$, which reduces the stress tensor space σ_{ij} to 3 equations of the plasticity condition.

2. Let us consider the axisymmetric state of an ideal isotropic incompressible rigid-plastic body in the cylindrical coordinate system ρ, θ, z. Omitting the intermediate calculations, we note that in the case of a smooth yield condition (1.4) the equation for determining the characteristic directions will have the form

$$f_3^2 \beta_3 \sin^2 2(\varphi + \psi) - \Phi \cos^2 2(\varphi + \psi) = 0, \tag{2.1}$$

where φ is the angle between the first principal direction and the ρ-axis; ψ is the angle between the characteristic of the first family in the ρz-plane and the ρ-axis. Since $\Phi < 0$, equation (2.1) will be satisfied only when

$$\psi = \varphi \pm \pi/4, \qquad f_3 = 0, \qquad f_1 + f_2 = 0. \tag{2.2}$$

Thus, lines of weak discontinuities on smooth yield surfaces can occur in the case when the state of stress in the ρz-plane represents simple shear; the lines of discontinuity coincide with the lines of maximum tangential stresses.

In the case when the axisymmetric stressed and deformed state corresponds to the edges (1.16), the two equilibrium equations and the two conditions (1.7) form a closed system of equations with respect to the four unknowns $\sigma_\rho, \sigma_\theta, \sigma_z, \tau_{\rho z}$.

We write the plasticity condition (1.4) in the form

$$\tau = \tau(p), \qquad \sigma_3 = \sigma_3(p), \qquad p = \frac{1}{2}(\sigma_1 + \sigma_2), \qquad \tau = \frac{1}{2}(\sigma_1 - \sigma_2).$$

The equation of the lines of weak discontinuity in the plane has the form

$$dz/d\rho = \left(F_2 \sin 2\varphi \pm \sqrt{F_2^2 - F_1^2}\right) / (F_1 - F_2 \cos 2\varphi), \tag{2.3}$$

$$F_2 = f_3(g_1 - g_2) - g_3(f_1 - f_2), \qquad F_1 = f_3(g_1 + g_2) - g_3(f_1 + f_2).$$

The relations for the stress components along the lines (2.3) are written in the form

$$dp\,\frac{F_2 - F_1\cos 2\varphi \pm \sin 2\varphi\sqrt{F_2^2 - F_1^2}}{F_2\sin 2\varphi \pm \sqrt{F_2^2 - F_1^2}} + d\tau\,\frac{F_1 - F_2\cos 2\varphi}{F_2\sin 2\varphi \pm \sqrt{F_2^2 - F_1^2}} +$$

$$+ 2\tau\,d\varphi + \frac{\tau}{r}\,dz + \frac{(p-\sigma_3)(\cos 2\varphi\,dz + \sin 2\varphi\,dr)}{r} = 0. \qquad (2.4)$$

The equations for the displacement velocities have lines of weak discontinuity coinciding with (2.3).

For an incompressible material, the plasticity conditions are written in the form $\tau = \text{const}$, $\sigma_3 - \frac{1}{2}(\sigma_1 + \sigma_2) = \text{const}$. In this case the lines of weak discontinuity, as follows from (2.3), are always orthogonal, and the relations along them coincide with the corresponding relations for the edges of piecewise-linear yield conditions [4].

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Note: Figure translations are in progress. See original paper for figures.