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Abstract

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ON NECESSARY AND SUFFICIENT CONDITIONS FOR THE VALIDITY OF THE LIMITING AMPLITUDE PRINCIPLE

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1°. Let $\mathscr{L}$ be a semibounded operator in a Hilbert space. Consider the nonstationary problems

$$u_{tt} + \mathscr{L}u = f(x)e^{i\omega t}; \tag{1}$$

$$u\big|_{t=0} = 0, \qquad u_t\big|_{t=0} = 0; \tag{2}$$

$$u_{tt} + \mathscr{L}u = 0; \tag{1^1}$$

$$u\big|_{t=0} = 0, \qquad u_t\big|_{t=0} = g(x). \tag{1^2}$$

Below we formulate conditions on the spectrum of the operator $\mathscr{L}$ that are necessary and sufficient for the limiting amplitude principle to hold for the operator $\mathscr{L}$ in the form ([1])

$$\frac{1}{T}\int_0^T u(x,t)e^{-i\omega t}\,dt - v(x,\omega) = o(1) \quad \text{as } T \to \infty. \tag{3}$$

We shall assume that the resolvent $(\mathscr{L}-\lambda I)^{-1}$ of the operator $\mathscr{L}$ has the following properties:

A. The resolvent of the operator $\mathscr{L}$ is an integral operator with kernel $H(x,y,\lambda)$. The kernel is an analytic function of λ for $\operatorname{Im}\lambda \neq 0$. For $\operatorname{Im}\lambda = 0$ the function $H(x,y,\lambda+i\varepsilon)$ $(\varepsilon > 0,\ \lambda \geq 0)$ tends uniformly to its limiting values $H(x,y,\lambda)$, when x, y, λ vary in a bounded domain, $|x-y| \geq \delta$. As $|x-y| \to 0$,

$$|H(x,y,\lambda)| = O(1/|x-y|).$$

B. For $\operatorname{Re}\lambda < 0,\ \operatorname{Im}\lambda = 0$, the function $H(x,y,\lambda)$ has a finite number of poles.

Assumption B is sometimes replaced by the assumption

C. For $\operatorname{Re}\lambda < 0$, the function $H(x, y, \lambda)$ is analytic in λ.

If C is satisfied, then the operator $\mathcal{L}$ has no spectrum to the left of zero; consequently,

$$(\mathcal{L}u, u) > 0$$

(the equality sign is excluded by condition A).

D.

$$\left|\int_D H(x, y, -p^2) f(y)\, dy\right| < \frac{c}{1 + |p|^\gamma}, \qquad \operatorname{Re} p \geq 0, \quad \gamma > 0, \quad \operatorname{Im} p \to \infty$$

for smooth $f(y)$ decreasing sufficiently rapidly, so that the integral written converges absolutely and uniformly with respect to p^2, varying in a bounded domain $\operatorname{Re} p \geq 0$.

F.

$$\left|\int_D [H(x, y, \lambda + h) - H(x, y, \lambda)] f(y)\, dy\right| \xrightarrow[|h|\to 0]{} 0, \qquad \operatorname{Re}\lambda \geq 0.$$

Assumption A is sometimes replaced by the assumption

A$_1$. If the positive spectrum of the operator $\mathcal{L}$ is continuous, then the resolvent kernel has continuous limiting values as $\varepsilon \to 0$, $\operatorname{Im}\lambda = 0$, $\operatorname{Re}\lambda \geq 0$.

The content of the present note consists of the following theorems.

Theorem 1. *Suppose that assumptions A, C, D, F are satisfied. Then, for the solution of problem (1)–(2), estimate (3) is valid. The function $v(x, \omega)$ solves the stationary problem*

$$\mathcal{L}v - \omega^2 v = f(x), \tag{4}$$

$v(x, \omega)$ admits an analytic continuation into the half-plane $\operatorname{Im}\omega > 0$.

Theorem 2. *Suppose that assumptions B, D are satisfied. The operator $\mathcal{L}$ has no discrete negative spectrum if and only if the solution of problem (1^1)—(1^2) admits the estimate*

$$\left|\int_0^t u(x, \tau)\, d\tau\right| = O(e^{\varepsilon t}), \qquad t \to \infty, \tag{5}$$

where $\varepsilon > 0$ is arbitrarily small.

Theorem 3. *Suppose that assumptions C, D, A$_1$ are satisfied. Then, for the operator $\mathcal{L}$ to have no positive eigenvalues, it is necessary and sufficient that*

$$\left|\int_0^t u(x, \tau) e^{-i\lambda\tau}\, d\tau\right| = O(1), \qquad t \to \infty, \tag{6}$$

for all $\lambda \geqslant 0$. In order that the point $\lambda_0 \geqslant 0$ not be a point of the discrete spectrum, it is sufficient that (6) hold for $\lambda = \lambda_0$. (The function $u(x, \tau)$ is the solution of problem (1^1)—(1^2).)

Remark 1. In Theorems 2 and 3 it is assumed that, as g, one may take any element from a set dense in the Hilbert space.

Theorem 4. *Suppose that assumptions B, D, A_1, F are satisfied. In order that the limiting amplitude principle in the form (3) hold for the operator $\mathcal{L}$, it is necessary and sufficient that assumptions A and C hold.*

Theorems analogous to Theorems 1 and 2 were obtained earlier by the author in ([1]) for the case $\mathcal{L} = -\Delta + p(x)$.

2^0. Let D be an infinite domain with smooth boundary Γ in three-dimensional space. Consider the integral operator

$$Kf = \int_D K(x, y) f(y)\, dy. \tag{7}$$

Theorem 5. *Suppose*

$$\iint_{D\,D} \left[|D_{xK}(x, y)|^2 + |D_{yK}(x, y)|^2 + p^2(x)|K(x, y)|^2 \right] \, dx\, dy < \infty, \tag{8}$$

where

$$p(x) > C(1 + |x|)^{6+a}, \qquad a > 0. \tag{9}$$

Then K is a nuclear operator (the definition of a nuclear operator is given in ([2])).

We use Theorem 5 to prove the following theorem:

Theorem 6. *Let D be a plane domain with boundary Γ, asymptotically approaching the boundary Γ_0 of an angle D_0, so that*

$$\rho(s, \Gamma) < \frac{C}{1 + |s|^{7+\alpha}}, \qquad \alpha > 0, \tag{10}$$

where $\rho(s, \Gamma)$ is the distance from the point $s \subset \Gamma$ to Γ_0.

Suppose

$$\mathcal{L}u = -\Delta u + c(x) u, \tag{11}$$

where

$$p^2(x)|c(x)| < \frac{c}{1+|x|^{2+b}}, \qquad b > 0. \tag{12}$$

Denote by $\mathcal{L}$ the operator of the Dirichlet problem for the differential expression (11) in the domain D, and by $\mathcal{L}_0$ the operator of the Dirichlet problem in the domain D_0 for the Laplace operator.

Under the assumptions made, there exist wave operators $W_\pm(\mathcal{L}, \mathcal{L}_0)$ and the scattering matrix $S = W_+^*(\mathcal{L}, \mathcal{L}_0)W_-(\mathcal{L}, \mathcal{L}_0)$.

For definitions pertaining to the notion of wave operators, see ([3]). The resolvent kernel of the Schrödinger operator for the Dirichlet problem in the domain D was studied in ([1]) (see also ([4,5])).

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.