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Abstract

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MATHEMATICS

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ASYMPTOTICS OF SOLUTIONS OF LINEAR SYSTEMS WITH SMALL PERTURBATIONS

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A system $dx/dt = A(t)x$ and a function $g(t) \geq 0$ are given. Theorem 1 gives necessary and sufficient conditions under which the solutions of the system

$$dx/dt = A(t)x \tag{I}$$

and the solutions of every system

$$dx/dt = A(t)x + \varphi(x, t), \tag{II}$$

where $\varphi(0, t) \equiv 0$ and $\varphi(x, t)$ has Lipschitz constant $g(t)$, have the same asymptotics.

Theorems 2-5 solve other problems (arising in stability theory). In various special cases, as corollaries, some known results of R. E. Vinograd, B. F. Bylov, D. M. Grobman, Hartman, Wintner, and Onuchic are obtained ([3–14]). Some of these corollaries are given here.

Let $e_1, e_2, \dots, e_n$ be a basis in E^n; $F(t)x_0$ a solution of system (I) equal to x_0 at $t = t_0$; $v_i(t) = |\operatorname{cosec} \alpha_i|$, where $\alpha_i(t)$ is the angle between $F(t)e_i$ and the hyperplane spanned by the vectors $F(t)e_k$ $(k \neq i)$; $v(t) = \max_{i=1,2,\dots,n} v_i(t)$.

Theorem 1. Suppose that for some t_0 and all $t \geq t_0$

$$\left| \int^t g(\tau) v_i(\tau) \frac{\|x(\tau)\|}{\|F(\tau)e_i\|} \, d\tau \right| \leq C_1 \frac{\|x(t)\|}{\|F(t)e_i\|} \quad (i = 1, 2, \dots, n), \tag{A}$$

where $C_1 < 1/2n$; $x(t)$ is an arbitrary solution of system (I) (in the lower limit t_0, if the right-hand side of the inequality $\to \infty$ as $t \to \infty$, and ∞ otherwise).

Then for each solution $x(t)$ of system (I) there is a solution $y(t) = x(t) + z(t)$ of system (II) such that

$$\|z(t)\| \leq q\|x(t)\|, \quad \text{where } q < 1, \tag{1}$$

and every solution $y(t)$ of system (II) is representable in this form, and between the initial data $x(t)$ and $y(t)$ there exists a homeomorphism.

Suppose that (A) is not satisfied for any C_1. Then for every solution $x(t)$ of system (I) there is a matrix function $B(t)$, $\|B(t)\| \leq g(t)$, such that for y of the system $dx/dt = A(t)x + B(t)x$ there does not exist a solution $x(t) + z(t)$ satisfying (1).

Remark. In the subsequent theorems the necessary conditions are not given, but they are as close to sufficient as here.

Proof of Theorem 1. By the substitution

$$x(t) = \sum_{i=1}^{n} \frac{\xi_i(t)}{\|F(t)e_i\|} F(t)e_i$$

system (II) is reduced to the form

$$d\xi_i/dt = p_i(t)\xi_i + \psi_i(\xi_1, \dots, \xi_n, t), \tag{III}$$

where

$$p_i(t) = \frac{d}{dt} \ln \|F(t)e_i\|;$$

$$|\psi_i(\xi^{(1)}, t) - \psi_i(\xi^{(2)}, t)| \leq \nu_i(t)g(t)\|x^{(1)} - x^{(2)}\|.$$

Fix $x(t)$ (in the new coordinates $\xi(t)$)–a solution of system (I). Then a solution of system (II) $x(t) + z(t)$ (in the new coordinates $\xi(t) + \zeta(t)$) is found from the system

$$\zeta_i(t) = \int \exp\left[\int_{\tau}^{t} p_i(\tau)\, d\tau\right] \psi_i(\xi(\tau) + \zeta(\tau), \tau)\, d\tau \qquad (i = 1, 2, \dots, n) \tag{2}$$

(the lower limit in the integral is the same as in (A)).

By means of Tikhonov' s principle it is proved that there exists a solution $z(t)$ of this system satisfying (1). To each solution $x(t)$ of system (I) we associate a solution of system (II) $y(t) = x(t) + z(t)$, where $z(t)$ satisfies (1). It remains to prove that all solutions of system (II) are obtained in this way, i.e. that the multivalued mapping $I+\varphi : x(t_0) \to x(t_0) + z(t_0)$ is a mapping of E^n onto E^n. For this purpose we single out from $I+\varphi$ a homeomorphism $\Phi = I+\bar{\varphi}$ and prove that $\Phi(E^n)$ is closed in E^n. Since $\Phi(E^n)$ is nonempty and open in E^n (see (2), Ch. 5, proposition [3:14]), it follows that $\Phi(E^n) = E^n$. The homeomorphism Φ is constructed by transfinite induction.

Well-order E^n: $x_0 = 0,\ x_1,\ x_2, \dots,\ x_\omega, \dots$. Put $z_0 = \bar{\varphi}(x_0) = 0$. Suppose that a single-valued mapping $z_\alpha = \bar{\varphi}(x_\alpha)$ has been defined for $\alpha < \beta$ such that

$$\|z_\alpha(t) - z_{\alpha'}(t)\| \le q\|x_\alpha(t) - x_{\alpha'}(t)\| \qquad \text{for } \alpha, \alpha' < \beta.$$

Then it is proved that the set of functions $z(t)$ such that

$$\|z(t) - z_\alpha(t)\| \le q\|x_\beta(t) - x_\alpha(t)\| \qquad (\alpha < \beta),$$

is nonempty, and by means of Tikhonov' s principle it is proved that in it there exists a solution of system (2) $z_\beta(t)$. We put $z_\beta(t_0) = z_\beta = \bar{\varphi}(x_\beta)$. By transfinite induction a contracting mapping $\bar{\varphi}$ has been constructed. Then $\Phi = I + \bar{\varphi}$ is a homeomorphism and $\Phi(E^n)$ is closed in E^n.

Corollary 1. If $A(t) \equiv A$, then $\nu_i(t) \le Ct^{m-1}$, where m is the maximum order of the Jordan blocks of A, and condition (A) becomes

$$\int^\infty \tau^{m-1} g(\tau)\, d\tau$$

(see (9, 11)).

Corollary 2. If

$$\int_t^\infty \nu(\tau) g(\tau)\, d\tau < \infty; \tag{*}$$

$$\int_\tau^t (p_{i+1} - p_i)\, d\tau \ge -d; \qquad \int^\infty (p_{i+1} - p_i)\, d\tau = \infty, \tag{**}$$

then condition (A) is fulfilled.

In R. E. Vinograd' s theorem (4) the stronger condition

$$A(t) = \begin{pmatrix} p_1(t) & 0 & \\ & \ddots & \\ 0 & & p_n(t) \end{pmatrix}; \qquad \int_\tau^t (p_{i+1} - p_i)\, d\tau \ge a(t-\tau) - d \quad (a > 0)$$

gives only preservation of the characteristic exponents (and not exact asymptotics) under perturbation.

Theorem 2. Let, for every $\varepsilon > 0$, there be a $t_0(\varepsilon)$ such that

$$\left| \int \nu_i(\tau) g(\tau) \frac{\|x(\tau)\|}{\|F(\tau)e_i\|} e^{\varepsilon\tau}\, d\tau \right| \leq \frac{q_\varepsilon}{n} \frac{\|x(t)\|}{\|F(t)e_i\|} e^{\varepsilon t} \qquad (i = 1, 2, \dots, n)$$

($q_\varepsilon < 1$ for all $\varepsilon > 0$, $q_{\varepsilon_0} < 1/2$ for some $\varepsilon_0 > 0$) for any solution $x(t)$ of system (I) for all $t \geq t_0(\varepsilon)$ (the lower limit of the integral is $t_0(\varepsilon)$, if the right-hand side of the inequality $\to \infty$ as $t \to \infty$, and ∞ otherwise).

Then there exists a homeomorphism between the initial data of the solutions $x(t)$ of system (I) and the initial data of the solutions $y(t)$ of system (II), under which, for any $\varepsilon > 0$,

$$\|y(t)\| \leq C_\varepsilon e^{\varepsilon t} \|x(t)\|$$

for the corresponding solutions.

Proof is analogous to the proof of Theorem 1, with the necessary complications.

Corollary 3. If $v(t)g(t) \to 0$ as $t \to \infty$ and $(**)$ is satisfied, then the conditions of Theorem 2 are satisfied.

Thus, under perturbations the characteristic exponents do not increase under conditions broader than the conditions of R. E. Vinograd [4], under which the characteristic exponents are preserved.

Theorem 3. Suppose that, for all solutions $x(t)$ of system (I), $\|x(t)\| \leq C_{x(t)} R(t)$. Suppose there exists t_0, a function $C(t) \leq K$, with $C(t_0)R(t_0) < 1/n$,

$$\left| \int^t g(\tau) v_i(\tau) \frac{\|x(\tau)\| + C(\tau)R(\tau)}{\|F(\tau)e_i\|}\, d\tau \right| \leq \frac{C(t)R(t)}{\|F(t)e_i\|} \qquad (i = 1, 2, \dots, n)$$

(the lower limit is t_0, if the right-hand side of the inequality $\to \infty$ as $t \to \infty$, and ∞ otherwise) for $t \geq t_0$ and for all solutions $x(t)$ of system (I) such that $\|x(t_0)\| = 1$.

Then, if $y(t)$ is a solution of system (II), then

$$\|y(t)\| \leq C_{y(t)} R(t).$$

Theorem 4. Suppose system (I) is stable and the conditions of Theorem 3 are satisfied, with $R(t) \equiv 1$. Then system (II) is also stable.

Theorem 5. Suppose that, for all solutions $x(t)$ of system (I),

$$\|x(t)\| \leq C_{\varepsilon,x(t)} R(t) e^{\varepsilon t}$$

for all $\varepsilon > 0$. Suppose there exist t_0 and functions $C_\varepsilon(t) \leq K_\varepsilon$, with

$$C_{\varepsilon_0}(t_0) R(t_0) e^{\varepsilon_0 t_0} < 1/n$$

for some $\varepsilon_0 > 0$, and, for all $\varepsilon > 0$, $i = 1, 2, \dots, n$,

$$\left| \int^t g(\tau) v_i(\tau) \frac{\|x(\tau)\| + C_\varepsilon(\tau) R(\tau) e^{\varepsilon\tau}}{\|F(\tau) e_i\|} \, d\tau \right| \leq \frac{C_\varepsilon(t) R(t) e^{\varepsilon t}}{\|F(t) e_i\|}$$

(the lower limit is t_0, if the right-hand side of the inequality $\to \infty$ as $t \to \infty$, and ∞ otherwise) for $t \geq t_0$ and for all solutions $x(t)$ of system (I) for which $\|x(t_0)\| = 1$.

Then, if $y(t)$ is a solution of system (II), then

$$\|y(t)\| \leq C_{\varepsilon,y(t)} R(t) e^{\varepsilon t} \qquad \text{for all } \varepsilon > 0.$$

Theorem 4 follows from Theorem 3, and the proofs of Theorems 3 and 5 are entirely analogous to the proofs of Theorems 1 and 2, respectively.

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CITED LITERATURE

1. A. N. Tychonoff, Math. Ann., **3**, No. 5, 767 (1935).
2. P. S. Aleksandrov, *Combinatorial Topology*, Moscow–Leningrad, 1947.
3. R. E. Vinograd, Doctoral dissertation, Moscow State University, 1958.
4. R. E. Vinograd, DAN, **119**, No. 4 (1958).
5. R. E. Vinograd, DAN, **114**, No. 3 (1957).
6. R. E. Vinograd, Mat. sborn., **42**, issue 2 (1957).
7. B. F. Bylov, Prikl. matem. i mekh., **14**, issue 4 (1950).

8. B. F. Bylov, Dissertation, Moscow State University, 1954.

9. D. M. Grobman, DAN, **108**, No. 4 (1956).

10. D. M. Grobman, Matem. sborn., **30** (72), issue 1 (1952).

11. P. Hartman, A. Wintner, Am. J. Math., **77**, 45 (1955).

12. P. Hartman, N. Onuchic, Pacific J. Math., **13**, No. 4, 1193 (1963).

13. N. Onuchic, Michigan Math. J., **11**, No. 3, 237 (1964).

14. N. Onuchic, J. Math. Anal. and Appl., **6**, No. 3, 457 (1963).

Note: Figure translations are in progress. See original paper for figures.