A Probabilistic Description of the Random Process of Motion of a Rarefied Gas

1965

SovietRxiv

Abstract

Full Text

AERODYNAMICS

S. V. Vallander

A Probabilistic Description of the Random Process of Motion of a Rarefied Gas

(Presented by Academician L. I. Sedov, 24 XI 1964)

At the present time, for rarefied gases there are well-known (see, for example, (1-3)) kinetic equations from which one determines the densities f_i of the mathematical expectation of the number of particles in the phase space $(\mathbf{r}, \mathbf{v})$ of coordinates and velocities. Equally well known are (4, 5) formulations of problems on determining the densities of the mathematical expectations $f_i(\mathbf{r}, \mathbf{v}, t)$. However, an exhaustive description of the random process of motion of a rarefied gas is not completed by finding the functions f_i, and for the description of this random process it is also necessary to construct a number of probability-distribution densities.

The purpose of the present work is to indicate a certain necessary set of probability densities, to give formulas for their calculation, and thereby to supplement the description of the random processes of motion of rarefied gases.

In order not to distract attention with secondary difficulties, in the present work we have in mind a simple gas consisting of structureless particles. The transfer of the results obtained to gas mixtures of particles with internal degrees of freedom presents no difficulty.

1. Let us first consider a homogeneous gas and denote by $P_n(V)$ the probability of detecting n particles of this gas in a volume V. Under the assumptions, natural for a rarefied gas, of independence of random events, to determine the functions $P_n(V)$ we have the system of recurrence equations

$$P_n(V_1 + V_2) = \sum_{i=0}^{n} P_{n-i}(V_1) P_i(V_2). \tag{1}$$

The functions $P_n(V)$ must satisfy the natural conditions

$$P_0(0) \neq 0; \tag{2}$$

$$\lim_{V \to 0} P_n(V)/V^n < \infty; \tag{3}$$

$$\sum_{n=0}^{\infty} P_n(V) = 1. \tag{4}$$

Under conditions (2), (3), and (4), the solution of the system of functional equations (1) exists, is unique, and has the form

$$P_n(V) = \frac{(\mu V)^n}{n!} e^{-\mu V}. \tag{5}$$

The constant μ in (5) has the meaning of the mean volume density of the number of particles.

2. For the Poisson distribution (5) the addition theorem is well known. Therefore the passage from formulas (5) for a homogeneous gas to more general formulas for an inhomogeneous gas is carried out without any difficulty. If $\mu(\mathbf{r}, t)$ is the density of the mathematical expectation of the number of particles

in an inhomogeneous gas, then from (5) for an inhomogeneous gas we obtain the formulas

$$P_n(V) = \exp\left[-\iiint\limits_{(V)} \mu\, dV\right] \left(\iiint\limits_{(V)} \mu\, dV\right)^n \Big/ n!. \tag{6}$$

3. From formula (6) (since they are suitable for all volumes), by simple reasoning we obtain formulas for the densities of the distribution of the coordinates of n gas particles in the volume V. For the density $\varphi_n(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n, t \mid V)$ we obtain the formula

$$\varphi_n(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n, t \mid V) = \exp\left[-\iiint\limits_{(V)} \mu\, dV\right] \prod_{i=0}^{n} \mu(\mathbf{r}_i, t)/n!. \tag{7}$$

From (7), obviously, (6) follows immediately if the corresponding integration is carried out.

It is also essential that the probability densities found pertain to the volume V. From (7) we see that universal probability densities of the distribution of coordinates (without specifying the volume V) do not exist.

4. Let us introduce into consideration the function $\nu(\mathbf{r}, \mathbf{v}, t)$ of the velocity distribution of a gas particle that is certainly located at the point with radius vector $\mathbf{r}$ at the time t.

We shall have

$$\iiint_{-\infty}^{+\infty} \nu(\mathbf{r}, \mathbf{v}, t)\, d\omega = 1, \tag{8}$$

where $d\omega$ is the volume element in velocity space. Let us also introduce into consideration the density of the mathematical expectation of the number of particles $f(\mathbf{r}, \mathbf{v}, t)$ in the phase space $(\mathbf{r}, \mathbf{v})$ of coordinates and velocities.

We have

$$f(\mathbf{r}, \mathbf{v}, t) = \mu(\mathbf{r}, t)\nu(\mathbf{r}, \mathbf{v}, t). \tag{9}$$

From (9) and (8) follow the formulas

$$\mu(\mathbf{r}, t) = \iiint_{-\infty}^{+\infty} f(\mathbf{r}, \mathbf{v}, t)\, d\omega, \tag{10}$$

$$\nu(\mathbf{r}, \mathbf{v}, t) = f(\mathbf{r}, \mathbf{v}, t) \Big/ \iiint_{-\infty}^{+\infty} f(\mathbf{r}, \mathbf{v}, t)\, d\omega, \tag{11}$$

which make it possible to find the two functions $\mu(\mathbf{r}, t)$ and $\nu(\mathbf{r}, \mathbf{v}, t)$, if only one distribution function $f(\mathbf{r}, \mathbf{v}, t)$ is known.

5. Formulas (7) make it possible (under obvious assumptions of independence) to write formulas for the densities

$$\pi_n(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n, t \mid V) \tag{12}$$

of the joint distribution of the coordinates of n gas particles in the spatial volume V and their velocities in the entire velocity space.

Obviously, we have

$$\begin{aligned} &\pi_n(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n, \mathbf{v}_1, \dots, \mathbf{v}_n, t \mid V) = \\ &= \exp\left[-\iiint\limits_{(V)} \mu\, dV\right] \prod_{i=1}^{n} \mu(\mathbf{r}_i, t)\nu(\mathbf{r}_i, \mathbf{v}_i, t) \big/ n!. \end{aligned} \tag{13}$$

Using (9) and (10), we can express all the densities π_n through only one distribution function f. Obviously, we have

$$\pi_n(\mathbf{r}_1, \dots, \mathbf{v}_n, t \mid V) = \exp\left[-\iiint\limits_{(V)} \iiint_{-\infty}^{+\infty} f(\mathbf{r}, \mathbf{v}, t)\, dV\, d\omega\right] \prod_{i=1}^{n} f(\mathbf{r}_i, \mathbf{v}_i, t) \Big/ n!. \tag{14}$$

Formulas (14), written for $n = 1, 2, \dots, k, \dots$, give, in a certain sense, an exhaustive probabilistic description of the random process of motion of a rarefied gas, if the distribution function $f(\mathbf{r}, \mathbf{v}, t)$ is known. These formulas, in particular, make it possible to solve all questions concerning the determination of the probability distributions of the values of aggregate quantities of interest to the physicist and the aerodynamicist, such, for example, as density, amount of motion, or energy. If formulas (14) are used for homogeneous gases, they give probabilistic characteristics that somewhat supplement the known results of statistical physics. Naturally, for large volumes, by virtue of the limit theorems of probability theory, we return to the usual results of statistical physics concerning mean quantities.

The distribution function entering formulas (14) should be sought as the solution of the corresponding kinetic equation. In the case of a mixture of gases with internal degrees of freedom, one should deal with a system of distribution functions f_i.

Leningrad State University
named after A. A. Zhdanov

Received
18 XI 1964

CITED LITERATURE

[1] L. Boltzmann, *Lectures on Gas Theory*, Moscow, 1956.
[2] S. Chapman, T. Cowling, *The Mathematical Theory of Non-Uniform Gases*, IL, 1960.
[3] S. V. Vallander, DAN, **131**, No. 1 (1960).
[4] S. V. Vallander, *Proceedings of the All-Union Congress on Theoretical and Applied Mechanics*, Publishing House of the Academy of Sciences of the USSR, 1962.
[5] S. V. Vallander, *Aerodynamics of Rarefied Gases*, Collection 1, Leningrad, 1963.

Note: Figure translations are in progress. See original paper for figures.