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Abstract

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CONSTRUCTIVE LOCALLY CONVEX LINEAR TOPOLOGICAL SPACES

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1. In the present note all special terms and notation not explained are understood as in ($^{1-3}$), and all assertions are understood in the sense of the constructive interpretation (^{2}).

Let: a) A_m and A_n be alphabets; b) $\mathfrak{P}$ be a one-parameter formula in a variable α of type t_m; c) $\mathfrak{E}$ be a two-parameter formula in variables β and γ of type t_m; d) $+$ be an algorithm of type $(\theta\theta \to \theta)$ in the alphabet A_m^{ca}, where θ is a subordinate generic letter whose characteristic formula is $\mathfrak{P}$; e) $\cdot$ be an algorithm of type $(p\theta \to \theta)$ in the alphabet $A_m^{ca} \cup \Psi_2$; f) $\mathfrak{D}$ be a fixed word of type θ; g) $\mathfrak{J}$ be a **normal** one-parameter formula in a variable δ of type t_n, defining a nonempty set of words of type t_n; h) $\mathfrak{L}$ be a two-parameter formula in variables ζ of type t_n and η of type t_m.

We shall agree to denote: by ι, the subordinate generic letter whose characteristic formula is $\mathfrak{J}$; by $\theta_1, \theta_2, \ldots$, variables of type θ; by $\iota_1, \iota_2, \ldots$, variables of type ι. Introduce the notation:

$$(T = U) \leftrightarrows F^{\beta,\gamma}_{T,U}[\mathfrak{E}]; \qquad \mathfrak{L}(V, U) \leftrightarrows F^{\zeta,\eta}_{V,U}[\mathfrak{L}];$$

$$(T + U) \leftrightarrows +(T\Box U); \qquad a \cdot T \leftrightarrows \cdot(a\Box T),$$

where T and U are arbitrary terms of type θ; V is an arbitrary term of type ι; a is any rational number.

The list

$$A_m,\ \mathfrak{P},\ \mathfrak{E},\ +,\ \cdot,\ \mathfrak{D},\ A_n,\ \mathfrak{J},\ \mathfrak{L} \qquad (1)$$

will be called a **constructive locally convex linear topological space** if the following conditions are satisfied:

I 1. $\forall\theta_1(\theta_1 = \theta_1)$.

I 2. $\widetilde{\forall}(\theta_1 = \theta_2 \;\&\; \theta_1 = \theta_3 \supset \theta_2 = \theta_3)$.

II 1. $\widetilde{\forall}(\theta_1 = \theta_2 \;\&\; \theta_3 = \theta_4 \supset \theta_1 + \theta_3 = \theta_2 + \theta_4)$.

II 2. $\widetilde{\forall}(a = b \;\&\; \theta_1 = \theta_2 \supset a \cdot \theta_1 = b \cdot \theta_2)$.

II 3. $\widetilde{\forall}(\theta_1 + \theta_2 = \theta_2 + \theta_1)$.

II 4. $\widetilde{\forall}(\theta_1 + (\theta_2 + \theta_3) = (\theta_1 + \theta_2) + \theta_3)$.

II 5. $\forall \theta_1 (0 \cdot \theta_1 = \mathfrak{D})$.

II 6. $\forall \theta_1 (1 \cdot \theta_1 = \theta_1)$.

II 7. $\widetilde{\forall}(a \cdot (b \cdot \theta_1) = (a \cdot b) \cdot \theta_1)$.

II 8. $\widetilde{\forall}((a + b) \cdot \theta_1 = a \cdot \theta_1 + b \cdot \theta_1)$.

II 9. $\widetilde{\forall}(a \cdot (\theta_1 + \theta_2) = a \cdot \theta_1 + a \cdot \theta_2)$.

III 1. $\forall \iota_1 \theta_1 \theta_2 (\theta_1 = \theta_2 \ \& \ \mathfrak{L}(\iota_1, \theta_1) \supset \mathfrak{L}(\iota_1, \theta_2))$.

III 2. $\forall \iota_1 \iota_2 \exists \iota_3 \forall \theta_1 (\mathfrak{L}(\iota_3, \theta_1) \supset \mathfrak{L}(\iota_1, \theta_1) \ \& \ \mathfrak{L}(\iota_2, \theta_1))$.

III 3. $\forall \iota_1 \theta_1 \theta_2 ab (M(a) + M(b) \doteq 1 \ \& \ \mathfrak{L}(\iota_1, \theta_1) \ \& \ \mathfrak{L}(\iota_1, \theta_2) \supset \mathfrak{L}(\iota_1, a \cdot \theta_1 + b \cdot \theta_2))$.

III 4. $\forall l_1 \theta_1 \exists a \, (a > 0 \,\&\, \mathfrak{D}(l_1, a \cdot \theta_1))$.

III 5. $\forall l_1 \exists l_2 \forall \theta_1 \theta_2 \, (\mathfrak{D}(l_2, \theta_1) \,\&\, \mathfrak{D}(l_2, \theta_2) \supset \mathfrak{D}(l_1, \theta_1 + \theta_2))$.

Here a, b are variables for rational numbers, and M is an algorithm for computing the absolute value of rational numbers.

Words of type θ are called points of the space (1); for each word l_1 of type l, the formula $\mathfrak{D}(l_1, \theta_1)$ determines a set of points of the space, which we shall call a neighborhood of zero with index l_1. A list $(A_n, \mathfrak{I}, \mathfrak{D})$ satisfying III1—III5 is called a locally convex topology of the space. A list $(A_m, \mathfrak{P}, \mathfrak{C}, +, \cdot, \mathfrak{D})$ satisfying I1—I9 is called a constructive linear space. Thus, a constructive locally convex space is a constructive linear space on which a locally convex topology is defined.

Remark. In the definition one should emphasize the assumption: $\mathfrak{I}$ must be a normal formula. This assumption, on the one hand, helps us prove a number of necessary properties of a constructive locally convex space; on the other hand, it does not restrict us in the further development of the theory.

It is not difficult to prove the following properties:

1) $\forall l_1 \theta_1 (\theta_1 = \mathfrak{D} \supset \mathfrak{D}(l_1, \theta_1))$;
2) $\forall l_1 \theta_1 a \, (M(a) \leqslant 1 \,\&\, \mathfrak{D}(l_1, \theta_1) \supset \mathfrak{D}(l_1, a \cdot \theta_1))$;
3) $\forall l_1 \exists l_2 \forall \theta_1 \theta_2 \theta_3 \theta_4 (\mathfrak{D}(l_2, \theta_3 - \theta_1) \,\&\, \mathfrak{D}(l_2, \theta_4 - \theta_2) \supset \mathfrak{D}(l_1, (\theta_3 + \theta_4) - (\theta_1 + \theta_2)))$;
4) $\forall a l_1 \exists l_2 \forall \theta_1 \theta_2 (\mathfrak{D}(l_2, \theta_2 - \theta_1) \supset \mathfrak{D}(l_1, a \cdot \theta_2 - a \cdot \theta_1))$;
5) $\forall a \theta_1 l_1 \exists n l_2 \forall b \theta_2 (M(b - a) < 2^{-n} \,\&\, \mathfrak{D}(l_2, \theta_2 - \theta_1) \supset \mathfrak{D}(l_1, b \cdot \theta_2 - a \cdot \theta_1))$,

where $(T - U) \overset{5}{=} (T + (-1) \cdot U)$ for any terms T, U of type θ.

Property 3) means the uniform continuity of addition; properties 4) and 5) mean the uniform continuity of multiplication by a fixed rational number and the continuity of multiplication in both arguments.

Let $(A_n, \mathfrak{I}, \mathfrak{D})$ and $(\widetilde{A}_n, \tilde{\mathfrak{I}}, \widetilde{\mathfrak{D}})$ be locally convex topologies defined on the linear space $(A_m, \mathfrak{P}, \mathfrak{C}, +, \cdot, \mathfrak{D})$. We shall say that the topology $(A_n, \mathfrak{I}, \mathfrak{D})$ majorizes the topology $(\widetilde{A}_n, \tilde{\mathfrak{I}}, \widetilde{\mathfrak{D}})$ if

$$\forall \tilde{l}_1 \exists l_1 \forall \theta_1 (\mathfrak{D}(l_1, \theta_1) \supset\sim \widetilde{\mathfrak{D}}(\tilde{l}_1, \theta_1)),$$

where $\tilde{l}_1$ is a variable for words of the set $\tilde{\mathfrak{I}}$. Two topologies are called equivalent if each of them majorizes the other.

2. Let $\mathfrak{A}$ be a subset of the set $\mathfrak{P}$. The set $\mathfrak{A}$ is called -closed in the space (1) if

$$\forall \theta_1 (\neg(\theta_1 \in \mathfrak{A}) \supset \exists l_1 \forall \theta_2 (\mathfrak{D}(l_1, \theta_2 - \theta_1) \supset \neg(\theta_2 \in \mathfrak{A})));$$

it is called ′-closed in the space (1) if

$$\forall \theta_1 (\forall l_1 \exists \theta_2 ((\theta_2 \in \mathfrak{A}) \,\&\, \mathfrak{D}(l_1, \theta_2 - \theta_1)) \supset (\theta_1 \in \mathfrak{A})).$$

Theorem 1. *There exists a constructive locally convex space in which there is a set that is -closed but not ′-closed.*

Theorem 2. *There exists a constructive locally convex space in which there is a set that is ′-closed but not -closed.*

For the proof of Theorem 1 it is enough to take the space of real duplexes with the usual topology and the set of those duplexes x such that $(x \leqslant 0 \vee x \geqslant 0)$. For the proof of Theorem 2 we shall consider the space $\mathfrak{S}$, formed from the linear space of real duplexes, on which a locally convex topology $(\mathfrak{Y}_0, \mathfrak{I}, \mathfrak{D})$ is defined, and moreover

$\mathscr{J}$ is the set of natural numbers and the formula $\mathfrak{D}$ is defined so that

$$\mathfrak{D}(i, x) \equiv \exists j \forall kl\, (k, l \geq j \supset M(x) \cdot M(S(k) - S(l)) < 2^{-i}),$$

where S is Specker’ s algorithm (see (6) or (3), § 8). In the space $\mathfrak{S}$ the set of such duplexes x that $M(x) \leq 1$ is $\mathscr{J}'$-closed, but not $\mathscr{J}$-closed.

3. Denote by $\mathfrak{M}$ the space (1). $\mathfrak{M}$ is called T-separable if

$$\forall \theta_1 (\neg(\theta_1 = \mathfrak{D}) \supset \exists l_1 \neg \mathfrak{D}(l_1, \theta_1)); \tag{T}$$

it is called T'-separable if

$$\forall \theta_1 (\forall l_1 \mathfrak{D}(l_1, \theta_1) \supset \theta_1 = \mathfrak{D}). \tag{T'}$$

It is easy to see that if $\mathfrak{M}$ is T-separable and $\mathcal{E}$ is a normal formula, then $\mathfrak{M}$ is also T'-separable. In general, these notions are not equivalent; namely, the following holds:

Theorem 3. *There exists a constructive locally convex space that is T'-separable but not T-separable.*

This is the space $\mathfrak{S}$ constructed in the proof of Theorem 2.

In the further development of the theory it will be useful to retain both axioms (T) and (T′) as constructive analogues of the separability axiom of classical topology.

4. A constructive locally convex space $\mathfrak{M}$ will be said to be a **space with multiplication by real duplexes** if, first, multiplication in $\mathfrak{M}$ is not only an algorithm of type $(p\theta \to \theta)$, but also an algorithm of type $(d\theta \to \theta)$, and if, second, after replacing in I1—I5 the variables a, b by variables x, y for duplexes, multiplication by rational numbers by multiplication by duplexes, and addition and multiplication of rational numbers by addition and multiplication of duplexes, we obtain true assertions.

It is not difficult to prove that if $\mathfrak{M}$ is a space with multiplication by duplexes, then after the indicated replacement in formulas 1)—5) we obtain true assertions. In this paragraph and in paragraph 5 we shall consider spaces with multiplication by real duplexes.

Of n points $\theta_1, \ldots, \theta_n$ in $\mathfrak{M}$ we shall say that they are **linearly independent** in $\mathfrak{M}$ if

$$\forall x_1 \ldots x_n (x_1 \cdot \theta_1 + \ldots + x_n \cdot \theta_n = \mathfrak{D} \supset x_1 = 0 \ \& \ \ldots \ \& \ x_n = 0).$$

The space $\mathfrak{M}$ is called **n-dimensional in the weak sense** if there are potentially realizable n linearly independent points $X_1, \ldots, X_n$ in $\mathfrak{M}$ such that

$$\forall \theta_1 \exists x_1 \ldots x_n (\theta_1 = x_1 \cdot X_1 + \ldots + x_n \cdot X_n).$$

The space $\mathfrak{M}$ is called **n-dimensional in the strong sense** if there are potentially realizable n linearly independent points $X_1, \ldots, X_n$ in $\mathfrak{M}$ and n algorithms $\lambda_1, \ldots, \lambda_n$ of type $(\theta \to d)$ such that

$$\forall \theta_1 (\theta_1 = \lambda_1(\theta_1) \cdot X_1 + \ldots + \lambda_n(\theta_1) \cdot X_n).$$

The space $\mathfrak{M}$ is called **infinite-dimensional** if, for any n, potentially realizable n linearly independent points in $\mathfrak{M}$ exist.

It is clear that if $\mathfrak{M}$ is n-dimensional in the strong sense, then it is n-dimensional in the weak sense. However, the converse assertion is not true; namely, the following holds:

Theorem 4. *For every natural number $n \geq 1$, there exists a locally convex space that is n-dimensional in the weak sense, but not n-dimensional in the strong sense.*

The proof of this theorem is based on the theorem that there is no algorithm transforming each F-number into an equal duplex (^{4}, ^{5}).

5. Along with the space $\mathfrak{M}$, let us consider the space $\breve{\mathfrak{M}}$, specified by the list

$$A_{\breve{\mathfrak{M}}},\ \breve{\mathfrak{P}},\ \breve{\mathfrak{C}},\ \breve{+},\ \breve{\cdot},\ \breve{\mathfrak{D}},\ A_{\breve{\mathfrak{M}}},\ \breve{\mathfrak{I}},\ \breve{\mathfrak{D}}. \tag{2}$$

All notations with the sign $\smile$ above for $\mathfrak{M}$ have meanings analogous to the meanings without the sign $\smile$ for $\mathfrak{M}$.

An algorithm λ of type $(\theta \to \breve{\theta})$ is called an **operator** of type $(\theta \to \breve{\theta})$ if

$$\forall \theta_1 \theta_2 (\theta_1 = \theta_2 \supset \lambda(\theta_1) \mathrel{\breve{=}} \lambda(\theta_2)).$$

An operator λ_2 of type $(\breve{\theta} \to \theta)$ is called the **inverse operator** to λ_1 of type $(\theta \to \breve{\theta})$ if

$$\forall \breve{\theta}_1 (\exists \theta_1 (\lambda_1(\theta_1) \mathrel{\breve{=}} \breve{\theta}_1) \supset \lambda_1(\lambda_2(\breve{\theta}_1)) \mathrel{\breve{=}} \breve{\theta}_1).$$

An operator λ of type $(\theta \to \breve{\theta})$ is called **one-to-one in both directions** if

$$\forall \theta_1 \theta_2 (\lambda(\theta_1) \mathrel{\breve{=}} \lambda(\theta_2) \supset \theta_1 = \theta_2);$$

it is called **linear** if

$$\forall \theta_1 \theta_2 (\lambda(\theta_1 + \theta_2) \mathrel{\breve{=}} \lambda(\theta_1) \mathbin{\breve{+}} \lambda(\theta_2)), \qquad \forall x \theta_1 (\lambda(x \cdot \theta_1) \mathrel{\breve{=}} \breve{x} \mathbin{\breve{\cdot}} \lambda(\theta_1)).$$

A linear operator λ of type $(\theta \to \breve{\theta})$ is called **continuous** if

$$\forall \breve{\iota}_1 \exists \iota_1 \forall \theta_1 (\mathfrak{D}(\iota_1, \theta_1) \supset \breve{\mathfrak{D}}(\breve{\iota}_1, \lambda(\theta_1))).$$

The spaces $\mathfrak{M}$ and $\breve{\mathfrak{M}}$ are called **isomorphic** if there are potentially realizable two operators λ_1 of type $(\theta \to \breve{\theta})$ and λ_2 of type $(\breve{\theta} \to \theta)$ such that they are one-to-one in both directions, linear, continuous, and each of them is inverse to the other.

Denote by R^n $(n \geq 1)$ the locally convex space formed from the linear space of all n-term τ-systems of duplexes, on which the topology $(\mathfrak{U}_0, \mathfrak{I}, \mathfrak{D})$ is defined, where $\mathfrak{I}$ is the set of natural numbers and for any point X of the form $x_1 \tau \ldots \tau x_n$

$$\mathfrak{D}(i, X) \equiv (\max(x_1 \square \ldots \square x_n) < 2^{-i}).$$

Theorem 5. For every n ($n \geq 1$) there exists a locally convex space that is n-dimensional in the strong sense and T-separable, but not isomorphic to the space R^n.

Theorem 6. Every one-dimensional in the strong sense and T-separable space is isomorphic to the space R^1.

One can prove further that if $\mathfrak{M}$ is n-dimensional in the strong sense, T-separable, and in it every $\mathfrak{Z}'$-closed set is $\mathfrak{Z}$-closed, then $\mathfrak{M}$ is isomorphic to R^n.

Theorem 5 is proved with the aid of the space $\mathfrak{S}$.

6. In the theory of constructive locally convex spaces we have constructed such structures as a subspace, a factor space, the completion of a space, the product of a family of spaces, the projective limit and the inductive limit of a sequence of spaces. Questions of separability and completeness in these structures were considered. It was also proved that there exists a family of T-separable spaces whose product is not T-separable, but for lack of space we shall not set out these questions here.

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Note: Figure translations are in progress. See original paper for figures.