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**Abstract**

**Full Text**

**MATHEMATICS**

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## ON LIMITS IN MEASURE OF INDETERMINACY AND LIMIT FUNCTIONS OF TRIGONOMETRIC AND ORTHOGONAL SERIES

It is known that trigonometric series and series with respect to complete orthonormal systems in many cases behave in exactly the same way as series

$$\Sigma \equiv \sum_{n=0}^{\infty} u_n(x), \quad (1)$$

whose terms are arbitrary measurable functions (see <sup>(1-3)</sup>).

In the present note we shall point out some further properties common both to series with arbitrary measurable terms and to trigonometric series, as well as to series with respect to complete orthonormal systems.

In what follows, instead of a series of the form (1), we shall consider sequences of measurable functions

$$f_m(x) \quad (m = 0, 1, 2, \dots), \quad (2)$$

defined almost everywhere on some segment  $[a, b]^*$ .

First of all, let us introduce some definitions and notation. We shall denote, respectively, by

$$\overline{\lim}_{m \rightarrow \infty} (\text{mes}, [a, b]) f_m(x), \quad \underline{\lim}_{m \rightarrow \infty} (\text{mes}, [a, b]) f_m(x) \quad (3)$$

the upper and lower limits in measure on  $[a, b]$  of the sequence (2)\*\*. In what follows, the upper and lower limits in measure on  $[a, b]$  of the series (1) will mean, respectively, the upper and lower limits in measure on  $[a, b]$  of the sequence of partial sums of this series. We shall denote the upper and lower limits in measure on  $[a, b]$  of the series (1), respectively, by

$$\overline{\Sigma}(\text{mes}, [a, b]), \quad \underline{\Sigma}(\text{mes}, [a, b]). \quad (4)$$

Let us also introduce the following definition. Functions  $g(x)$  and  $h(x)$ , defined almost everywhere on some segment  $[a, b]$ , will be called, respectively, the **upper** and **lower limits in measure on  $[a, b]$  in the narrow sense** of the sequence (2), if for every increasing sequence of natural numbers

$$m_k \quad (k = 0, 1, 2, \dots) \tag{5}$$

the equalities

$$g(x) = \overline{\lim}_{k \rightarrow \infty} (\text{mes}, [a, b]) f_{m_k}(x),$$

$$h(x) = \underline{\lim}_{k \rightarrow \infty} (\text{mes}, [a, b]) f_{m_k}(x). \tag{6}$$

hold.

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\* The functions  $f_m(x)$  may be equal to  $+\infty$  or  $-\infty$  on sets of positive measure, whereas the terms  $u_n(x)$  of the series (1) must be finite almost everywhere on the segment on which the series (1) is considered.

\*\* The definition of the upper and lower limit in measure of a sequence of measurable functions is given in <sup>(1)</sup> (p. 4) and <sup>(2)</sup> (p. 294).

We shall denote the upper and lower limits in measure on  $[a, b]$ , in the narrow sense, of the sequence (2), respectively, by

$$\overline{\lim}_{m \rightarrow \infty} (\text{mes}, [a, b])^+ f_m(x), \quad \underline{\lim}_{m \rightarrow \infty} (\text{mes}, [a, b])^+ f_m(x). \tag{7}$$

If  $S_m(x)$  ( $m = 0, 1, 2, \dots$ ) are the partial sums of the series (1), then we shall call, respectively, the upper and lower limits in measure on  $[a, b]$ , in the narrow sense, of the series (1) the functions

$$\overline{\lim}_{m \rightarrow \infty} (\text{mes}, [a, b])^+ S_m(x), \quad \underline{\lim}_{m \rightarrow \infty} (\text{mes}, [a, b])^+ S_m(x), \tag{8}$$

and we shall denote these functions by

$$\overline{\Sigma}(\text{mes}, [a, b])^+, \quad \underline{\Sigma}(\text{mes}, [a, b])^+. \tag{9}$$

Assuming that the sequence (2) has the same properties as before, denote by  $M[f_m(x)]$  the set of all pairs of functions  $g(x), h(x)$  for which one can define an increasing sequence of natural numbers (5) such that the equalities (6) will hold. If  $\Sigma$  is a series of the form (1), then we put  $M[\Sigma] = M[S_m(x)]$ , where  $S_m(x)$  are the partial sums of the series  $\Sigma$ .

Further, we denote by  $M^+[f_m(x)]$  the set of all pairs of functions  $g(x), h(x)$  for which one can choose an increasing sequence of natural numbers (5) such that the equalities

$$g(x) = \overline{\lim}_{k \rightarrow \infty} (\text{mes}, [a, b])^+ f_{m_k}(x),$$

$$h(x) = \underline{\lim}_{k \rightarrow \infty} (\text{mes}, [a, b])^+ f_{m_k}(x). \quad (10)$$

will hold.

If  $\Sigma$  is a series of the form (1), then we put  $M^+(\Sigma) = M^+[S_m(x)]$ , where  $S_m(x)$  are the partial sums of the series  $\Sigma$ .

As is known, complete orthonormal systems of functions on some segment  $[a, b]$  are a special case of normalized bases in the space  $\mathcal{L}^p[a, b]$ , where  $p > 1$ .<sup>\*</sup> If  $\{\psi_\nu(x)\}$  ( $\nu = 0, 1, 2, \dots$ ) is a normalized basis in the space  $\mathcal{L}^p[a, b]$ , then by a series with respect to this basis we shall mean any series of the form

$$\Sigma \equiv \sum_{\nu=0}^{\infty} c_\nu \psi_\nu(x), \quad (11)$$

where  $c_\nu$  ( $\nu = 0, 1, 2, \dots$ ) are constants.

**Theorem 1.** *Take an arbitrary system (2) of measurable functions  $f_m(x)$ , defined almost everywhere on some segment  $[a, b]$ , and an arbitrary normalized basis  $\{\psi_\nu(x)\}$  ( $\nu = 0, 1, 2, \dots$ ) in the space  $\mathcal{L}^p[a, b]$ ,  $p > 1$ .*

*Then one can define a series  $\Sigma$  (see (11)) with respect to this basis which has the following properties:*

a°.

$$\lim_{\nu \rightarrow \infty} c_\nu = 0; \quad (12)$$

b°.

$$M[f_m(x)] = M[\Sigma];$$

c°.

$$M^+[f_m(x)] \subset M^+[\Sigma].$$

Moreover, if  $[a, b] = [-\pi, \pi]$ , then one can define a trigonometric series

$$T \equiv \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \quad (13)$$

\* The definition of normalized bases in the space  $\mathcal{L}^p[a, b]$  may be found, for example, in (3), p. 353.

satisfying the conditions

a'.

$$\lim_{n \rightarrow \infty} a_n = 0, \quad \lim_{n \rightarrow \infty} b_n = 0; \quad (14)$$

b'.

$$M[f_m(x)] = M[T];$$

c'.

$$M^+[f_m(x)] \subset M^+[T].$$

Let us formulate one more theorem, which is proved with the aid of Theorem 1. First of all, let us recall the definition of a limiting function of some sequence of functions.

Let a sequence (2) of functions  $f_m(x)$  be given, defined almost everywhere on some segment  $[a, b]$ . We shall say that a function  $\varphi(x, E)$ , defined almost everywhere on a set  $E$ , is a limiting function of the sequence (2) if, first,

$$\text{mes } E > 0, \quad E \subset [a, b] \quad (15)$$

and, second, there exists an increasing sequence (5) of natural numbers  $m_k$  such that

$$\lim_{k \rightarrow \infty} f_{m_k}(x) = \varphi(x, E) \quad (16)$$

almost everywhere on  $E^*$ .

Further, a function  $\varphi(x, E)$ , defined almost everywhere on a set  $E$ , will be called a limiting function of the series (1) if conditions (15) are satisfied and, moreover,  $\varphi(x, E)$  is a limiting function of the partial sums of the series (1).

Let us introduce also the following definition. A function  $\varphi(x, E)$ , defined almost everywhere on a set  $E$ , will be called a limiting function in the strict sense of the sequence (2) if conditions (15) are fulfilled and if one can define such an increasing sequence (5) of natural numbers  $m_k$  that, first, equality (16) holds almost everywhere on  $E$ , and, second, the sequence of functions

$$f_{m_k}(x) \quad (k = 0, 1, 2, \dots) \quad (17)$$

does not tend to any limit (neither finite nor infinite of a definite sign) almost everywhere on the set  $[a, b] - E$ .

If  $\varphi(x, E)$  is a limiting function in the strict sense of the sequence of partial sums of the series (1), then we shall say that  $\varphi(x, E)$  is a limiting function in the strict sense of this series.

**Theorem 2.** *Suppose that  $M = \{\varphi(x, E)\}$  is the set of all limiting functions of the sequence (2) of measurable functions  $f_m(x)$ , defined almost everywhere on some segment  $[a, b]$ .*

*Then, for any normalized basis  $\{\psi_\nu(x)\}$  ( $\nu = 0, 1, 2, \dots$ ) in the space  $\mathcal{L}^p[a, b]$ ,  $p > 1$ , one can define a series (11) satisfying condition (12) and such that  $M$  is the set of all limiting functions of the series (11), and all these limiting functions are limiting functions in the strict sense of the series (11).*

*Moreover, if  $[a, b] = [-\pi, \pi]$ , then one can define a trigonometric series (13) satisfying conditions (14) and possessing the same properties as the series (1).*

Theorem 2 is a generalization of results proved in <sup>(2, 3)</sup>. In particular, Theorems 1 and 2 will be valid if in their formulations the system  $\{\psi_\nu(x)\}$  is a complete orthonormal system.

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## CITED LITERATURE

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- <sup>3</sup> A. Talalyan, Matem. sborn., **56(98)**, no. 3, 353 (1962).
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\* <sup>(4)</sup>, Definition 7, p. 275. In this definition the functions of the sequence (2) need not necessarily be assumed measurable.

*Note: Figure translations are in progress. See original paper for figures.*

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