ON THE EXTENSION OF PERFECT MAPPINGS TO \(\omega\alpha\)-EXTENSIONS

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MATHEMATICS

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ON THE EXTENSION OF PERFECT MAPPINGS TO $\omega\alpha$-EXTENSIONS

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E. G. Sklyarenko in the paper (^{1}) proved the following theorem:

Let $\bar{X}$ and $\bar{Y}$ be bicompact extensions of completely regular spaces X and Y, with $\bar{X}$ perfect and $\bar{Y}$ having a punctiform remainder. Every perfect mapping $f : X \to Y$ extends to a mapping

$$\bar{f} : \bar{X} \to \bar{Y}.$$

In the present paper this result is generalized to the case of T_1-spaces and $\omega\alpha$-extensions.

Lemma 1. *Let two T_1-spaces X and Y and a perfect mapping $f : X \to Y$ be given. Then the mapping f extends to a perfect mapping $\bar{f}$ of the space ωX onto ωY, taking the remainder $\omega X \setminus X$ into the remainder $\omega Y \setminus Y$.*

Proof. By Theorem 1.3 of A. V. Arkhangel' skii (^{2,3}), the mapping f extends (uniquely) to a continuous closed mapping $\bar{f}$ of the space ωX onto ωY. The mapping $\bar{f}$ takes the remainder $\omega X \setminus X$ into the remainder $\omega Y \setminus Y$. Indeed, suppose that for some point $\eta \in Y$ the preimage $\bar{f}^{-1}\eta$ contains at least one point $\xi \in \omega X \setminus X$. The set $f^{-1}\eta \subset X$ is closed in X and, by the condition of the lemma, bicompact. Since ωX is an extension, $[f^{-1}\eta]_{\omega X} = f^{-1}\eta$ (Definition 2 (^{4})). Consequently, the point ξ cannot be a limit point for $f^{-1}\eta$. But it also cannot be isolated, because, by the construction of the mapping $\bar{f}$ (Theorem 1 (^{2})), the point ξ must have as its coordinate the set $f^{-1}\eta$; in view of these circumstances the supposition that the preimage $\bar{f}^{-1}\eta$ contains points from $\omega X \setminus X$ is false. The lemma is proved.

Remark 1. In what follows, without making a special stipulation, we shall denote a point of the space of an upper semicontinuous decomposition by the same symbol as the body of the corresponding element of the decomposition, only in parentheses; for example, (A) is the point of the space of the upper semicontinuous decomposition corresponding to the element of the decomposition whose body is the set A.

Remark 2. Without making a special stipulation, by $\omega\alpha X$ we shall denote the $\omega\alpha$-extension of the space X (Definition 2 (^{4})).

Lemma 2. *For any centered and relatively maximal with respect to centeredness system* $\gamma = \{P_\beta\}$ *of closed sets of the space* X*, the set* $\bigcap_\beta [P_\beta]_{\omega\alpha X}$ *consists of one point.*

Proof. Let $\gamma = \{P_\beta\}$ be an arbitrary maximal centered system of closed sets of the space X. We shall prove that $\bigcap_\beta [P_\beta]_{\omega\alpha X}$ consists of one point.

First of all,

$$\bigcap_\beta [P_\beta]_{\omega\alpha X} \neq \Lambda$$

in view of the closedness of $[P_\beta]_{\omega\alpha X}$ and the bicompactness of $\omega\alpha X$. The sets of the system γ are coordinates of some point of the space ωX: $\xi = \{P_\beta\}$. We shall show that some point $(A) \in \omega\alpha X$ is a point of contact of the system γ in that if and only if $\xi \in A$. In one direction the assertion is obvious, i.e., if $\xi \in A$, then $(A) \in \bigcap_\beta [P_\beta]_{\omega\alpha X}$. Conversely, if $\xi \in A$, then $(A) \in \bigcap_\beta [P_\beta]_{\omega\alpha X}$. Indeed, if $\xi \in A$, then for an arbitrary point $\xi_A \in A$ there is a set $P_{\beta_1} \in \xi$ such that $\xi_A \in [P_{\beta_1}]_{\omega X}$; hence the point ξ_A has a neighborhood $O\xi_A$ not meeting the set P_{β_1}. The neighborhoods $\{O\xi_A\}$ chosen in this way form a cover of the set A, from which, by the bicompactness of A, one can choose a finite cover $\{O\xi_{A_i}\}$. Put $OA = \bigcup_{i=1}^k O\xi_{A_i} \supset A$; $P_A = \bigcap_{i=1}^k P_{\beta_i}$, where from $P_{\beta_i} \in \xi$ it follows that $P_A \in \xi$. Since $P_{\beta_i} \cap O\xi_{A_i} = \Lambda$, we have $P_A \cap OA = \Lambda$; hence $(A) \in [P_A]_{\omega\alpha X}$, and consequently $(A) \in \bigcap_\beta [P_\beta]_{\omega\alpha X}$. But there exists one and only one element of the partition containing ξ; denote it by A_ξ. Hence, by what was proved above, it follows that

$$(A_\xi) = \bigcap_\beta [P_\beta]_{\omega\alpha X}.$$

Remark. For all points $\xi = \{P_\beta\}$ we have

$$\bigcap_\beta [P_\beta]_{\omega\alpha X} = (A).$$

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* A continuous mapping is called perfect if it is closed and the full preimages of all points are bicompact.

Lemma 3. Let f be a continuous mapping of the space $\omega\alpha X$ onto $\omega\alpha Y$. If the mapping f on X is perfect and $fX = Y$, then the relation

$$f(\omega\alpha X \setminus X) = \omega\alpha Y \setminus Y$$

holds.

Proof. The mapping $f : \omega\alpha X \xrightarrow{\text{onto}} \omega\alpha Y$ generates a certain mapping $g = f\varphi$ of the space ωX onto $\omega\alpha Y$, where φ is the natural mapping of the space ωX onto $\omega\alpha X$, which exists by Definition 2 (^{4}); the mapping f induces a certain mapping f_0 on X, which, by the hypothesis of the lemma, is perfect. On the basis of Lemma 1, f_0 extends (uniquely) to a perfect mapping $\bar{f}_0$ of the space ωX onto ωY, carrying the remainder $\omega X \setminus X$ into the remainder $\omega Y \setminus Y$.

The mapping $\bar{f}_0$ generates a certain continuous mapping g_0 of the space ωX onto $\omega\alpha Y$, which is defined as follows:

$$g_0\xi = \bar{f}_0\xi = f_0\xi, \qquad \text{if } \xi \in X;$$

$$g_0\xi = (B), \qquad \text{if } \bar{f}_0\xi \in B \text{ or } g_0 = \psi\bar{f}_0,$$

where ψ is the natural mapping of the space ωY onto $\omega\alpha Y$, which exists by Definition 2 (^{4}). We obtain the diagram:

$$\begin{tikzcd}
\omega X \arrow[r, "\bar{f}_0"] \arrow[d, "\varphi"'] & \omega Y \arrow[d, "\psi"] \\
\omega\alpha X \arrow[r, "f"] & \omega\alpha Y
\end{tikzcd}$$

Let us prove that this diagram is commutative, i.e., that the mapping $g_0 = \psi\bar{f}_0$ coincides with the mapping $g = f\varphi$. At the points of the space X this is obvious. Let $\xi \in \omega X \setminus X$. The maximal centered system of closed sets $\xi = \{P_\alpha\}$ of the space X is carried, under the mapping f_0, into a maximal centered system of closed sets $\eta = \{f_0 P_\alpha\}$ of the space Y. The centeredness of the system η is obvious. We shall show its maximality. Suppose the contrary, i.e., suppose that the centered system η is not maximal; then there exists such a ⋯

a closed set $F \subset Y$, $F \in \eta$, upon adding which to the system η we again obtain a centered system. Hence it follows that for an arbitrary finite subsystem of closed sets $P_{\alpha_1}, P_{\alpha_2}, \dots, P_{\alpha_k}$ of the system $\xi = \{P_\alpha\}$ we have

$$f_0 P_{\alpha_1} \cap f_0 P_{\alpha_2} \cap \cdots \cap f_0 P_{\alpha_k} \cap F \neq \Lambda$$

or else

$$P_{\alpha_1} \cap P_{\alpha_2} \cap \cdots \cap P_{\alpha_k} \cap f_0^{-1} F \neq \Lambda,$$

where $f_0^{-1} F \in \xi$. Thus the system ξ, together with the set $f_0^{-1} F$, again forms a centered system, contrary to the maximality of ξ.

The contradiction obtained proves the maximality of the system η. By Theorem 2 (^{4}), $\omega\alpha Y$ is the space of some continuous external decomposition of the space ωY; hence the point $\eta \in \omega Y$ belongs to some class $B \subset \omega Y$. By the definition of f_0 (see Theorem 1 (^{2})), $\bar{f}_0\xi = \{f_0 P_\alpha\}$; consequently, $\{f_0 P_\alpha\} = \bar{f}_0\xi \in B$, and hence, by the definition of g_0, we have $g_0\xi = (B)$. Let $g\xi = (B')$; we shall prove that $(B') = (B)$, thereby proving the coincidence of the mappings g and g_0.

As we saw above, $\bar{f}_0\xi = \{f_0 P_\alpha\} \in B$; hence, on the basis of Lemma 2 (^{4}),

$$\bigcap_\alpha [f_0 P_\alpha]_{\omega\alpha Y} = (B).$$

But $\xi = \bigcap_\alpha [P_\alpha]_{\omega X}$. By Lemma 1 (^{4}), the mapping g is closed, and moreover

$$g[P_\alpha]_{\omega X} = [gP_\alpha]_{\omega\alpha Y} = [f_0 P_\alpha]_{\omega\alpha Y}$$

(since g coincides with f_0 on X); consequently,

$$g\xi = (B') \in \bigcap_\alpha [f_0 P_\alpha]_{\omega\alpha Y} = (B), \quad \text{i.e. } (B') = (B).$$

Thus it has been proved that the mappings g and g_0 coincide. On the basis of Lemma 1,

$$\bar{f}_0(\omega X \setminus X) = \omega Y \setminus Y;$$

hence it follows that

$$g_0(\omega X \setminus X) = \omega\alpha Y \setminus Y.$$

Since $f = g_0\varphi^{-1}$ and the mapping g coincides with the mapping g_0, we have

$$f(\omega\alpha X \setminus X) = \omega\alpha Y \setminus Y.$$

The lemma is completely proved.

For the case of completely regular spaces X and Y and their Čech extensions $\beta X, \beta Y$, this lemma was proved in (^{5}) (Lemma 1.5).

Definition. The $\omega\alpha$-extension $\omega\alpha X$ of a T_1-space X is called **perfect** if the natural mapping of the Wallman extension ωX onto $\omega\alpha X$ is monotone.

Theorem. Let the extension $\omega\alpha X$ be perfect, and let the extension $\omega\alpha Y$ have a punctiform* growth. Every perfect mapping

$$f : X \longrightarrow Y$$

extends to a perfect mapping

$$\tilde{f} : \omega\alpha X \longrightarrow \omega\alpha Y.$$

Proof. By Lemma 1, the mapping f extends to a perfect mapping $\bar{f}$ of the space ωX onto the space ωY, carrying the growth $\omega X \setminus X$ onto the growth

$\omega Y \setminus Y$. Since $\omega\alpha X$ is perfect by the hypothesis of the theorem, the natural mapping

$$\varphi : \omega X \longrightarrow \omega\alpha X$$

is monotone. By definition 2 (^{4}), there exists a natural mapping

$$\psi : \omega Y \longrightarrow \omega\alpha Y.$$

Let the point $(A) \in \omega\alpha X \setminus X$; by the monotonicity of φ, the set

$$\varphi^{-1}(A) = A \subset \omega X \setminus X$$

is connected and bicompact. Since f is perfect, $\bar{f}A$ is a connected bicompact subset of the growth $\omega Y \setminus Y$ (Lemma 3). Hence, from the punctiformity of the growth $\omega\alpha Y \setminus Y$ it follows that

$$\psi\bar{f}A = (B) \in \omega\alpha Y \setminus Y,$$

and thereby the single-valuedness of the mapping

$$\tilde{f} = \psi\bar{f}$$

is proved.

We shall show that

$$\tilde{f}\,\omega\alpha X = \omega\alpha Y.$$

Suppose the contrary; then no point is mapped to some point $(B_0) \in \omega\alpha Y \setminus Y$. By the punctiformity of the growth $\omega\alpha Y \setminus Y$, either the set $\bar{f}A$ is entirely contained in some element of the decomposition $B \subset \omega Y \setminus Y$, or it has no common points with this element of the decomposition. Therefore, for any point $(A) \in \omega\alpha X \setminus X$ we have

$$\bar{f}A \cap B_0 = \Lambda.$$

In other words, no point of the growth $\omega X \setminus X$ is mapped onto the set B—a contradiction to the fact that

$$\bar{f}\,\omega X = \omega Y.$$

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[*] A space is called punctiform if every connected bicompact subset of it consists of a single point.

The mapping $\tilde{f}$ is continuous. Indeed, take an arbitrary point $(A_0) \in \omega\alpha X$ and an arbitrary neighborhood V of the point $\tilde{f}(A_0)$; since the mappings ψ and $\bar{f}$ are continuous, the set $\bar{f}^{-1}\psi^{-1}V$ is open in the space ωX. From $\tilde{f}(A_0) \in V$ it follows that $A_0 \subseteq \bar{f}^{-1}\psi^{-1}(\tilde{f}(A_0)) \subseteq \bar{f}^{-1}\psi^{-1}V$; consequently, $A_0 \subset \bar{f}^{-1}\psi^{-1}V$.

By Theorem 2 (^{4}), $\omega\alpha X$ is a continuous external partition of the space ωX; consequently, in the neighborhood $\bar{f}^{-1}\psi^{-1}V$ of the element of the partition

A_0 there exists a marked neighborhood GA_0—the set corresponding to it in the space $\omega\alpha X$ will be a neighborhood of the point (A_0), which we denote by $U(A_0)$. By the construction of the neighborhood $U(A_0)$ we have

$$\tilde{f}U(A_0) = \psi\bar{f}\varphi^{-1}U(A_0); \qquad \psi^{-1}U(A_0) = GA_0.$$

By the definition of the neighborhood $U(A_0)$ (GA_0 is a marked neighborhood), $GA_0 \subseteq \bar{f}^{-1}\psi^{-1}V$, hence $\bar{f}GA_0 \subseteq \psi^{-1}V$, $\psi\bar{f}GA_0 \subseteq V$, i.e. $\psi\bar{f}\varphi^{-1}U(A_0) \subseteq V$. The continuity of the mapping $\tilde{f}$ is proved. Obviously, $\tilde{f}$ is an extension of the mapping f. On the basis of Theorem 7 (^{4}), $\tilde{f}$ is closed; consequently, it is perfect. The theorem is completely proved.

Remark 3. Any homeomorphism $h:\ X \xrightarrow{\text{onto}} Y$ (X, Y are T_1-spaces), by virtue of our theorem, extends to a perfect mapping $\bar{h}$ onto the perfect $\omega\alpha$-extensions of the spaces X and Y with punctiform remainder. It is not difficult to see that the mapping $\bar{h}$ is a homeomorphism of $\omega\alpha X$ onto $\omega\alpha Y$. Consequently, all perfect $\omega\alpha$-extensions with punctiform remainder (if they exist) of a T_1-space X coincide with one another up to homeomorphism.

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Note: Figure translations are in progress. See original paper for figures.

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