MATHEMATICS

D. F. DAVIDENKO

1965

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Abstract

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MATHEMATICS

D. F. DAVIDENKO

ON THE APPLICATION OF THE METHOD OF VARIATION OF A PARAMETER TO THE CONSTRUCTION OF ITERATIVE FORMULAS OF INCREASED ACCURACY FOR DETERMINING THE ELEMENTS OF AN INVERSE MATRIX

(Presented by Academician N. N. Bogolyubov, 12 XII 1964)

In the paper ([1]), for the inversion of an n-th order matrix $A(\lambda)$, whose elements are functions of a parameter λ taking prescribed values on some finite interval $\lambda_0 \leq \lambda \leq \lambda^*$, the method of variation of a parameter ([2]) is applied. In this case a differential equation is constructed which is satisfied by the inverse matrix $A^{-1}(\lambda)$, and it is integrated numerically on the prescribed interval $\lambda_0 \leq \lambda \leq \lambda^*$ by one of the methods of numerical integration of ordinary differential equations ([3,4]). Depending on the chosen method and the step of numerical integration, the elements of the sought matrix $A^{-1}(\lambda)$ for the prescribed values of the parameter λ are obtained approximately, with one or another degree of accuracy.

In the present paper, by the same method of variation of a parameter ([2]), iterative formulas of increased accuracy are constructed for refining approximate values of the elements of the inverse matrix $A^{-1}(\lambda)$ obtained by the method of variation of a parameter ([1]) (or by any other method). The iterative formulas obtained lead to the desired result even in those cases when the known iterative formula ([5]) does not give a convergent process.

For determining, with a prescribed accuracy, the elements of the inverse matrix $A^{-1}(\lambda)$, the method of variation of a parameter ([1]) can be combined with one of the iterative formulas proposed in the present paper and conveniently implemented on modern computers. In the case $n = 1$, the iterative formulas obtained can be effectively used for finding reciprocals on high-speed computers that do not have a division operation.

1°. Let $A(\lambda) = \|a_{ij}(\lambda)\|$ $(i, j = 1, 2, \dots, n)$ be a square matrix of order n, whose elements are functions of the parameter λ, taking prescribed values on a finite interval $\lambda_0 \leq \lambda \leq \lambda^*$. We shall call the matrix $X(\lambda) = \|x_{ij}(\lambda)\|$ $(i, j = 1, 2, \dots, n)$ inverse to the matrix $A(\lambda)$ if

$$A(\lambda)X(\lambda) - E = 0, \tag{1}$$

where E is the identity matrix.

Suppose that, for some prescribed value of the parameter λ, for example $\lambda = \bar{\lambda}$, approximate values of the elements of the inverse matrix $X(\lambda)$ have been obtained by the method of variation of a parameter ([1]) (or by some other method). We denote the approximate matrix obtained by $X_k(\bar{\lambda})$ and suppose that it is required to refine the elements of this matrix to a prescribed degree of accuracy.

For this purpose, using the method of variation of a parameter ([2]), we shall construct a series of iterative formulae of increased accuracy. Following the method of variation of a parameter, instead of (1) consider the following matrix equation:

$$A - X^{-1} = T^*, \tag{2}$$

where $T = \|t_{ij}\|$ $(i, j = 1, 2, \dots, n)$ is a parameter matrix. It is obvious that the solution of equation (2) for $T = 0$ is also a solution of equation (1).

The approximate solution X_k of equation (1) will be an exact solution of equation (2) for $T = T_k$, if T_k is determined by the formula

$$T_k = A - X_k^{-1}. \tag{3}$$

We shall further assume that the solution X of equation (2) is a function of the elements of the parameter matrix $T = \|t_{ij}\|$ $(i, j = 1, 2, \dots, n)$, i.e. $X = X(T)$. Let us differentiate equation (2) with respect to the parameter t_{ij} $(i, j = 1, 2, \dots, n)$. As a result we obtain

$$\frac{\partial X}{\partial t_{ij}} = X \frac{\partial T}{\partial t_{ij}} X \qquad (i, j = 1, 2, \dots, n). \tag{4}$$

The numerical values of the elements of the matrix $X(T)$ for $T = T_k$ are known to us:

$$\text{for } T = T_k \qquad X(T) = X_k. \tag{5}$$

In order to determine the values of the elements of the matrix $X(T)$ for $T = 0$, or, what is the same thing, the solution of equation (1), we numerically integrate equations (4) by one of the methods of numerical integration of ordinary differential equations ([3], [4]) on the matrix interval $[T_k, 0]$ with initial conditions (5). In doing so, we choose the step of numerical integration $H_k = \|h_{ij}^{(k)}\|$ $(i, j = 1, 2, \dots, n)$ equal to (3):

$$H_k = 0 - T_k = X_k^{-1} - A. \tag{6}$$

If the numerical values thus obtained for the elements of the matrix X_{k+1} still do not give, to the prescribed degree of accuracy, the inverse matrix for the matrix A, then the process described above may be repeated, and so on.

For each selected method of numerical integration, its own iterative process is obtained. The order of accuracy of the corresponding iterative formula will, generally speaking, be no higher than the order of accuracy of the selected method of numerical integration.

2°. Below we shall consider several methods of numerical integration of ordinary differential equations and obtain the corresponding iterative formulae.

1. Euler' s method. Suppose that Euler' s method ([3]) has been chosen for the numerical integration of equations (4). According to this method, the improved approximate values of the elements of the matrix X_{k+1} are found from the formula

$$X_{k+1} = X_k + \Delta X_k,$$

where the increment matrix ΔX_k is determined as follows:

$$\Delta X_k = \sum_{i,j=1}^{n} X_k \frac{\partial T}{\partial t_{ij}} X_k h_{ij}^{(k)} = X_{kH_kX}k.$$

Thus, for determining the elements of the inverse matrix X for the matrix A, we obtain the following iterative formula:

$$X_{k+1} = X_k(2E - AX_k), \qquad k = 0, 1, 2, ... \tag{7}$$

Formula (7) coincides with the known formula ([5]).

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Here and in what follows, instead of $A(\bar{\lambda})$, $X_\nu^{-1}(\bar{\lambda})$, and $X_\nu(\bar{\lambda})$, we shall write simply A, X_ν^{-1}, and X_ν.

2. The improved Euler-Cauchy method ([3]) leads to the following iterative formula:

$$X_{k+1} = X_k \left[E + \frac{1}{2}\Delta\tau_k(E + S_k)\right], \quad k = 0, 1, 2, \dots, \tag{8}$$

$$S_k = (E + \Delta\tau_k)^2, \qquad \Delta\tau_k = E - AX_k.$$

Here $\Delta\tau_{k+1}$ is expressed in terms of $\Delta\tau_k$ by the recurrence formula

$$\Delta\tau_{k+1} = \frac{1}{2}\Delta\tau_k^3(E + \Delta\tau_k), \quad k = 0, 1, 2, \dots .$$

Consequently, if $\Delta\tau_0$ satisfies the condition

$$\left\| \frac{1}{2}\Delta\tau_0^3(E + \Delta\tau_0) \right\| \le \rho < 1,$$

then the iterative process (8) converges to the matrix A^{-1}. In this case the error estimate can be obtained in the form

$$\|X_m - A^{-1}\| \le \|X_0\| \left[1 + \frac{9}{2(1-\rho)} \right] \rho^{3^m - 1}.$$

If the approximation X_0 to A^{-1} is such that $\|\Delta\tau_0\| \le k < 1$, then the error estimate has the form

$$\|X_m - A^{-1}\| \le \|X_0\| \frac{k^{3^m}}{1-k}.$$

As a matrix norm it is convenient to take the first or the second norm ([5]), as the most easily computed.

3. The Runge–Kutta method ([3,6]). In this case the iterative formula has the form

$$X_{k+1} = X_k \left[E + \frac{1}{6}\left(\mathcal{L}_k^{(1)} + 2\mathcal{L}_k^{(2)} + 2\mathcal{L}_k^{(3)} + \mathcal{L}_k^{(4)} \right) \right], \quad k = 0, 1, 2, \dots, \tag{9}$$

where

$$\mathcal{L}_k^{(1)} = \Delta\tau_k, \quad \mathcal{L}_k^{(2)} = \left(E + \frac{1}{2}\mathcal{L}_k^{(1)} \right)^2 \Delta\tau_k,$$

$$\mathcal{L}_k^{(3)} = \left(E + \frac{1}{2}\mathcal{L}_k^{(2)} \right)^2 \Delta\tau_k, \quad \mathcal{L}_k^{(4)} = \left(E + \frac{1}{2}\mathcal{L}_k^{(3)} \right)^2 \Delta\tau_k, \quad \Delta\tau = E - AX_k.$$

By virtue of formula (9), we have

$$\Delta\tau_{k+1} = \frac{1}{4!}\Delta\tau_k^5 W_k, \quad k = 0, 1, 2, \dots;$$

$$W_k = 4\Delta\tau_k\mu_k^2 - \Delta\tau_k\left(E + \frac{1}{8}\Delta\tau_k\right) - 4\left(\mu_k - \frac{3}{2}E\right)\left(\mu_k - \frac{1}{2}E\right),$$

$$\mu_k = \left(E + \frac{1}{2}\Delta\tau_k\right)^2\left[E + \frac{1}{4}\Delta\tau_k\left(E + \frac{1}{2}\Delta\tau_k\right)^2\right].$$

Consequently, if $\Delta\tau_0$ satisfies the condition

$$\left\|\frac{1}{4!}\Delta\tau_0^5 W_k\right\| \le \rho < 1,$$

then the iterative process (9) converges to the matrix A^{-1}. The error estimate has the form

$$\|X_m - A^{-1}\| \le \|X_0\|\left[1 + \frac{1\,466\,631}{2^{13}(1-\rho)}\right]\rho^{5^m-1}.$$

If the initial approximation X_0 is chosen so that $\|\Delta\tau_0\| \le k < 1$, then the error estimate has the form

$$\|X_m - A^{-1}\| \le \|X_0\|\frac{k^{5^m}}{1-k}.$$

We shall not dwell here on other methods of numerical integration.

3°. **Example 1.** Let $A = 7$, and suppose that for the reciprocal $X = 1/A$ an approximate value $X_0 = 0.2855$ is given ($\Delta\tau_0 = -0.9985$). It is required to refine this value to 9 correct decimal places.

The first iteration by formula (8) gives $X_1 = 0.142963804$. The second iteration already gives the required result, $X_2 = 0.142857142$. To obtain the same result by formula (7), it is necessary to perform 14 iterations.

If the approximate value X_0 is taken equal to 0.2865 ($\Delta\tau_0 = -1.0055$), then by formula (8) again the second iteration gives the required result. The iterative process by formula (7) in this case already diverges.

With the initial condition $X_0 = 0.385$ ($\Delta\tau_0 = -1.695$), the fourth iteration by formula (9) gives the required result, whereas the iterative process by formula (8) already diverges for $X_0 = 0.364$ ($\Delta\tau_0 = -1.548$) and still converges (8 iterations) for $X_0 = 0.363$ ($\Delta\tau_0 = -1.541$).

Example 2. Let the matrix be given

$$A = \begin{Vmatrix} -1 & 1 & 0 & 0 & 0 \\ 0.25 & -1 & 0.25 & 0 & 0 \\ 0 & 0.25 & -1 & 0.25 & 0 \\ 0 & 0 & 0.25 & -1 & 0.25 \\ 0 & 0 & 0 & 0.25 & -1 \end{Vmatrix}.$$

Suppose that it is required to determine the elements of the inverse matrix $X = A^{-1}$ with accuracy to 9 correct significant digits.

As the initial matrix X_0 we take the matrix A. The third iteration by formula (9) gives the required result. The iterative process by formula (7) in this case diverges, while formula (8) leads on the fifth iteration to the required result.

If as the initial matrix X_0 one takes the matrix obtained at the first step by formula (9), then the fifth iteration by formula (7) leads to the required result, and by formula (8) the third iteration does.

With the initial approximation $X_0 = -1.65E$, the third iteration by formula (9) gives the required result. The iterative processes (7) and (8) in this case both diverge.

The effectiveness of the iterative formulas constructed by the method of variation of a parameter for the case of a single nonlinear equation $P(x) = 0$, using the improved Euler-Cauchy method and the Runge-Kutta method, was illustrated, at our suggestion, on a concrete example by V. S. Demidov (Moscow University, 1962).

Remark 1. Instead of equation (2), other equations may also be considered. Thus, for example, equation (2) may be taken with matrix T equal to $(1-t)[A-X_0^{-1}]$ $(0 \le t \le 1)$ or $t[A-X_0^{-1}]$ $(1 \le t \le 0)$. In these cases the iterative formulas obtained are the same (7), (8), and (9), but by a somewhat simpler route.

However, strictly speaking, the choice of equation is not a matter of indifference.

Remark 2. The method described above for constructing iterative formulas of increased accuracy can be applied directly also to other equations and problems, for example, to integral equations, differential and integro-differential equations, as well as to more general functional equations.

Institute of Atomic Energy
named after I. V. Kurchatov

Received
21 X 1964

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Note: Figure translations are in progress. See original paper for figures.