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# BLAGOVEST SENDOV

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**Abstract**

**Full Text**

**MATHEMATICS**

BLAGOVEST SENDOV

## LINEAR METHODS OF APPROXIMATION OF PERIODIC FUNCTIONS WITH RESPECT TO A METRIC OF HAUSDORFF TYPE

*(Presented by Academician A. N. Kolmogorov, 10 VIII 1964)*

In this note some estimates are given for the Hausdorff distance for certain classical approximation apparatuses, such as Fejér sums, Jackson polynomials, and Vallée-Poussin integrals.

1. At the beginning we define the notion of the modulus of nonmonotonicity  $\mu_f(\delta)$  of a function  $f(x)$  as follows:

$$\mu_f(\delta) = \sup_{|x_1 - x_2| \leq \delta} \left\{ \sup_{x_1 \leq x \leq x_2} [ |f(x_1) - f(x)| + |f(x_2) - f(x)| ] - |f(x_1) - f(x_2)| \right\}.$$

The modulus of nonmonotonicity  $\mu(\delta)$ , with respect to the Hausdorff distance, plays a role analogous to that of the modulus of continuity with respect to the uniform distance.

2. Let  $B_{2\pi}$  denote the class of  $2\pi$ -periodic functions  $f(x)$  for which: a)  $\lim_{\delta \rightarrow +0} \mu_f(\delta) = 0$ , and b)  $|f(x)| \leq B$ .

It is not difficult to prove that if  $f(x) \in B_{2\pi}$ , then for every  $x$  there exist  $f(x-0)$  and  $f(x+0)$ , and  $f(x)$  lies between  $f(x-0)$  and  $f(x+0)$ .

3. Let  $f(x)$  be a given function. Denote by  $\bar{f}$  a point set; we shall call it the supplementary graph of  $f(x)$ , which is obtained as the intersection of all closed point sets  $F$  in the plane containing the graph of the function  $f(x)$  and convex with respect to the  $y$ -axis (i.e., every straight line parallel to the  $y$ -axis intersects  $F$  in a point or in a segment).

It is not difficult to see that if  $f(x) \in B_{2\pi}$ , then  $\bar{f}$  consists of all points  $X(x, y)$  of the plane for which  $y$  lies between  $f(x-0)$  and  $f(x+0)$ , while if  $f(x)$  is a continuous function, then  $\bar{f}$  coincides with the graph of the function  $f(x)$ .

As in <sup>(1-3)</sup>, we define the distance  $r(f, g)$  between two functions  $f(x)$  and  $g(x)$  as follows:

$$r(f, g) = \max \left\{ \max_{X \in \bar{f}} \min_{Y \in \bar{g}} \|X - Y\|_0, \max_{X \in \bar{g}} \min_{Y \in \bar{f}} \|X - Y\|_0 \right\},$$

where

$$\|X - Y\|_0 = \|X(x_1, y_1) - Y(x_2, y_2)\|_0 = \max[|x_1 - x_2|, |y_1 - y_2|].$$

Such a distance between point sets was considered by F. Hausdorff <sup>(4)</sup>. Some properties of the distance  $r(f, g)$  with respect to continuous functions can be found in <sup>(1)</sup>.

In working with the distance  $r(f, g)$ , the following simple lemma is useful.

**Lemma 1.** *If: a) for every point  $X \in \bar{f}$  there exists a point  $Y \in \bar{g}$  such that  $\|X - Y\|_0 \leq \delta$ ; b) for every point  $X \in \bar{g}$  there exists a point  $Y \in \bar{f}$  such that  $\|X - Y\|_0 \leq \delta$ , then  $r(f, g) \leq \delta$ .*

4. Let  $K(t)$  be a function defined on the interval  $[-\pi, \pi]$ , with integrable square, for which: a)  $K(t) \geq 0$ ; b)

$$\int_{-\pi}^{\pi} K(t) dt = 1$$

and c)  $K(t) = K(-t)$ . If  $f(x) \in B_{2\pi}$ , then by  $\varphi(t) = \varphi_f(t)$  we denote the function

$$\varphi(x) = \int_{-\pi}^{\pi} f(x+t)K(t) dt. \quad (1)$$

**Theorem 1.** If  $f(x) \in B_{2\pi}$  and  $\varphi(x)$  is defined by equality (1), then

$$r(f, g) \leq \inf_{\delta \geq 0} \max \left[ \delta, \frac{1}{2} \mu(4\delta) + 4B \int_{\delta}^{\pi} K(t) dt \right],$$

where  $\mu(\delta)$  is the modulus of nonmonotonicity of  $f(x)$ .

The proof of the theorem, according to Lemma 1, follows from the following assertions.

If  $f(x) \in B_{2\pi}$  and has modulus of nonmonotonicity  $\mu(\delta)$ ,  $\varphi(x)$  is defined by equality (1), and  $\delta \geq 0$ , then:

- 1) For each point  $X(x_1, y_1) \in \bar{f}$  there exists a point  $Y(x_2, y_2) \in \bar{\varphi}$  such that

$$|x_1 - x_2| \leq \delta, \quad |y_1 - y_2| \leq \frac{1}{2} \mu(4\delta) + 2B \int_{\delta}^{\pi} K(t) dt.$$

2) For each point  $X(x_1, y_1) \in \bar{\varphi}$  there exists a point  $Y(x_2, y_2) \in \bar{f}$  such that

$$|x_1 - x_2| \leq \delta, \quad |y_1 - y_2| \leq 4B \int_{\delta}^{\pi} K(t) dt.$$

The latter assertions are obtained on the basis of the following property of functions from  $B_{2\pi}$ .

**Lemma 2.** Let  $f(x) \in B_{2\pi}$  and have modulus of nonmonotonicity  $\mu(\delta)$ ,  $x_0 = \frac{1}{2}(x_1 + x_2)$ ,  $x_1 < x_2$ , and let  $y_0$  be an arbitrary number between  $f(x_0 - 0)$  and  $f(x_0 + 0)$ . Then the inequality

$$f(x) \geq y_0 - \frac{1}{2}\mu(|x_1 - x_2|)$$

holds for all  $x \in [x_1, x_0]$  or for all  $x \in [x_0, x_2]$ , and the inequality

$$f(x) \leq y_0 + \frac{1}{2}\mu(|x_1 - x_2|)$$

holds for all  $x \in [x_1, x_0]$  or for all  $x \in [x_0, x_2]$ .

5. We give some consequences of Theorem 1.

Denote the  $n$ -th Fejér sum by

$$\sigma_n(x) = \frac{1}{2n\pi} \int_{-\pi}^{\pi} f(t) \left( \frac{\sin n(t-x)/2}{\sin(t-x)/2} \right)^2 dt = \frac{1}{2n\pi} \int_{-\pi}^{\pi} f(x+t) \left( \frac{\sin nt/2}{\sin t/2} \right)^2 dt,$$

the  $n$ -th Jackson polynomial by

$$U_n(x) = \frac{3}{2\pi n(2n^2 + 1)} \int_{-\pi}^{\pi} f(x+t) \left( \frac{\sin nt/2}{\sin t/2} \right)^4 dt;$$

the  $n$ -th Vallée-Poussin integral for a function  $f(x) \in B_{2\pi}$  with modulus of nonmonotonicity  $\mu(\delta)$  by

$$V_n(x) = \frac{(2n)!!}{(2n-1)!!} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+t) \cos^{2n} \frac{t}{2} dt.$$

From Theorem 1 the following estimates are obtained:

1)

$$r(f, \sigma_n) \leq \max \left[ \delta, \frac{1}{2}\mu(4\delta) + \frac{4B}{n\delta} \right],$$

$$r(f, U_n) \leq \max \left[ \delta, \frac{1}{2} \mu(4\delta) + \frac{10\pi^2 B}{(n\delta)^3} \right],$$

$$r(f, V_n) \leq \max \left[ \delta, \frac{1}{2} \mu(4\delta) + 4B\sqrt{\pi n} \left( 1 - \frac{\delta^2}{\pi^2} \right)^{2n} \right];$$

2) if  $\mu(\delta) \geq \delta$ ,

$$r(f, \sigma_n) \leq (1 + 16B)\mu(n^{-1/2}),$$

$$r(f, U_n) \leq (1 + 10\pi^2 B)\mu(n^{-3/4}),$$

$$r(f, V_n) \leq \mu(4\pi\sqrt{\ln n}/n) \quad \text{for } n \geq e^{8B^2/\pi};$$

3) if  $\mu(\delta) \leq \delta$ ,

$$r(f, \sigma_n) \leq 4\sqrt{2B}n^{-1/2},$$

$$r(f, U_n) \leq 12\sqrt[4]{B}n^{-3/4},$$

$$r(f, V_n) \leq 2\pi\sqrt{\ln n}/n \quad \text{for } n \geq e^{64B^2/\pi};$$

4) if  $\mu(\delta) \leq k\delta^\alpha$ , for  $0 < \alpha \leq 1$  and  $k > 0$ ,

$$r(f, \sigma_n) \leq 4\sqrt{2kB}n^{-\alpha/(1+\alpha)} + O(n^{-1/(1+\alpha)}),$$

$$r(f, U_n) \leq cn^{-3\alpha/(3+\alpha)}, \quad \text{where } c = \max[1, 2k + 10\pi^2 B].$$

It can be proved that the inequalities in item 4) cannot be improved in the sense of order.

Using the Poisson kernel, one can obtain the following theorem.

**Theorem 2.** Let  $f \in B_{2\pi}$  and

$$f(x) \sim \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx)$$

be its Fourier series. Denote

$$f_r(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} r^m (a_m \cos mx + b_m \sin mx);$$

then

$$r(f, f_r) \leq \inf_{\delta \geq 0} \max \left[ \delta, \frac{1}{2} \mu(4\delta) + \frac{2B(1-r^2)}{r^2 \delta} \right]$$

and, consequently,

$$\lim_{r \rightarrow 1-0} r(f, f_r) = 0.$$

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*Note: Figure translations are in progress. See original paper for figures.*

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