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ON THE SPECTRAL WIDTH OF THE RADIATION OF A QUANTUM GENERATOR

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In the study of quantum generators, the question of the spectral width of the radiation is of interest. The fact of spectral narrowing during generation is well known ([1]). The equations obtained in the present work for the dynamics of radiation in two-level systems have been used to calculate the spectral width of the radiation in a ruby generator. It is shown that the radiation spectrum narrows down to a certain limit, depending on the magnitude of the pumping and on the width of the spontaneous-emission spectrum.

Let us consider the problem of the interaction of a system of N_0 two-level atoms with the radiation field inside a resonator with ideally reflecting walls. The Hamiltonian of such a system has the form ([2], [3])

$$H = \sum_{j=1}^{N_0} \frac{\hbar\omega_0}{2}\sigma_z^j + \sum_{\mathbf{k}\lambda} \hbar\omega_k a^+_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} + \sum_{\mathbf{k}\lambda j} \hbar \left[B_{\mathbf{k}\lambda j} a_{\mathbf{k}\lambda}\sigma_+^j + B^*_{\mathbf{k}\lambda j} a^+_{\mathbf{k}\lambda}\sigma_-^j \right],$$

where

$$B_{\mathbf{k}\lambda j} = \sqrt{2\pi/\hbar\omega_{\mathbf{k}} V}\,(\mathbf{M}^j \mathbf{e}^\lambda_{\mathbf{k}})\, e^{i\mathbf{k}\mathbf{x}_j}.$$

The terms in H represent, respectively, the operators of the proper energy of the atoms, of the free quantized radiation field, and of the interaction of the atoms with the radiation field. The quantities $a_{\mathbf{k}\lambda}$ and $a^+_{\mathbf{k}\lambda}$ are the operators of annihilation and creation of a photon with momentum $\hbar\mathbf{k}$ and polarization $\mathbf{e}^\lambda_{\mathbf{k}}$. M^j and M^{j*} are the matrix elements of the transitions for an atom with coordinate $\mathbf{x}_j$. The operators σ_z^j, $\sigma_\pm^j$ satisfy the commutation relations

$$[\sigma_z^j, \sigma_\pm^{j'}] = \pm 2\sigma_\pm^j \delta_{jj'}, \qquad [\sigma_\pm^j, \sigma_\mp^{j'}] = \sigma_z^j \delta_{jj'}.$$

For simplicity we have used an expansion of the field in plane waves, which corresponds to a resonator in the form of a parallelepiped of volume V. Using the rules for differentiating operators, it is not difficult to obtain a system of coupled equations for the quantum-mechanical mean values:

$$i\frac{d}{dt}\langle a^{+}_{\mathbf{k}\lambda}a_{\mathbf{k}\lambda}\rangle = \langle\varphi_{\mathbf{k}\lambda}\rangle;$$

$$i\frac{d}{dt}\langle\varphi_{\mathbf{k}\lambda}\rangle = (\omega_0-\omega_k)\langle\psi_{\mathbf{k}\lambda}\rangle-\sum_{\mathbf{k}'\lambda'j}\left[B^{*}_{\mathbf{k}'\lambda'j}B_{\mathbf{k}\lambda j}\langle\sigma^{j}_{z}(a_{\mathbf{k}\lambda}a^{+}_{\mathbf{k}'\lambda'})\rangle + B_{\mathbf{k}\lambda j}B^{*}_{\mathbf{k}'\lambda'j}\langle\sigma^{j}_{z}(a_{\mathbf{k}\lambda}a^{+}_{\mathbf{k}'\lambda'})\rangle\right]-\sum_{jj'}\left[B^{*}_{\mathbf{k}\lambda j}B_{\mathbf{k}\lambda j'}\langle\sigma^{j'}_{+}\sigma \right. \tag{1}$$

$$i\frac{d}{dt}\langle\psi_{\mathbf{k}\lambda}\rangle = (\omega_0-\omega_k)\langle\varphi_{\mathbf{k}\lambda}\rangle-\sum_{\mathbf{k}'\lambda'j}\left[B_{\mathbf{k}'\lambda'j}B^{*}_{\mathbf{k}\lambda j}\langle\sigma^{j}_{z}(a_{\mathbf{k}\lambda}a^{+}_{\mathbf{k}'\lambda'})\rangle - B_{\mathbf{k}\lambda j}B^{*}_{\mathbf{k}'\lambda'j}\langle\sigma^{j}_{z}(a_{\mathbf{k}\lambda}a^{+}_{\mathbf{k}'\lambda'})\rangle\right]-\sum_{jj'}\left[B^{*}_{\mathbf{k}\lambda j}B_{\mathbf{k}\lambda j'}\langle\sigma^{j'}_{+}\sigma \right.$$

$$i\frac{d}{dt}\left\langle\sum_{j}\sigma^{j}_{z}\right\rangle = -2\sum_{\mathbf{k}\lambda}\langle\varphi_{\mathbf{k}\lambda}\rangle,$$

where

$$\varphi_{\mathbf{k}\lambda} \equiv \sum_{j}\left[B^{*}_{\mathbf{k}\lambda j}(a_{\mathbf{k}\lambda}\sigma^{j}_{-}) - B_{\mathbf{k}\lambda j}(a_{\mathbf{k}\lambda}\sigma^{j}_{+})\right],$$

$$\psi_{\mathbf{k}\lambda} \equiv \sum_{j}\left[B^{*}_{\mathbf{k}\lambda j}(a^{+}_{\mathbf{k}\lambda}\sigma^{j}_{-}) + B_{\mathbf{k}\lambda j}(a_{\mathbf{k}\lambda}\sigma^{j}_{+})\right],$$

$\langle a^{+}_{\mathbf{k}\lambda}a_{\mathbf{k}\lambda}\rangle \equiv n_{\mathbf{k}\lambda}$ is the mean number of photons with momentum $\hbar\mathbf{k}$ and polarization $\mathbf{e}^{\lambda}_{\mathbf{k}}$ in the volume V. In what follows, in equations (1) we shall omit terms nondiagonal in $\mathbf{k}, \mathbf{k}'$ and j, j'. These terms describe scattering of a photon by the j-th atom, as well as emission of a photon by one atom with its subsequent absorption by another atom. The smallness of the nondiagonal terms can be justified ([3]) by taking into account that the number of atoms in the volume is very large and that the wavelength of the radiation is small in comparison with the linear dimensions of the volume.

Since the coupling between atoms mediated by the radiation field is sufficiently weak, one may assume that only real transitions occur between the states of free particles and fields during the entire process of generator operation. Then the relations

$$\left\langle\sum_{j}\sigma^{j}_{z}(a^{+}_{\mathbf{k}\lambda}a_{\mathbf{k}\lambda}+a_{\mathbf{k}\lambda}a^{+}_{\mathbf{k}\lambda})\right\rangle = \left\langle\sum_{j}\sigma^{j}_{z}\right\rangle\langle a^{+}_{\mathbf{k}\lambda}a_{\mathbf{k}\lambda}+a_{\mathbf{k}\lambda}a^{+}_{\mathbf{k}\lambda}\rangle = N_{-}(2n_{\mathbf{k}\lambda}+1),$$

will be valid, where N is the population difference between the upper and lower atomic levels, $N_- = N_2 - N_1$. Next, into the system of equations obtained in this way we introduce the relaxation parameters γ_1, W_0, and γ_2, which describe, respectively, photon losses in the resonator, pumping of the crystal by an external source, which creates a positive population difference, and broadening of the upper level under the influence of vibrations of the crystal lattice. This system of equations takes the form

$$\dot{n}_{\mathbf{k}\lambda} + \gamma_1 n_{\mathbf{k}\lambda} = -\varphi_{1\mathbf{k}\lambda}, \qquad \dot{N}_- - W_0(N_0 - N_-) = 2\sum_{\mathbf{k}\lambda}\varphi_{1\mathbf{k}\lambda},$$

$$\dot{\varphi}_{1\mathbf{k}\lambda} + \frac{\gamma_2}{2}\varphi_{1\mathbf{k}\lambda} = (\omega_0 - \omega_{\mathbf{k}})\varphi_{2\mathbf{k}\lambda} - \frac{\omega_0}{2\tau_0^2\omega_{\mathbf{k}}} n_{\mathbf{k}\lambda}N_- - \frac{\omega_0}{2\tau_0^2\omega_{\mathbf{k}}} N_2, \tag{2}$$

$$\dot{\varphi}_{2\mathbf{k}\lambda} + \frac{\gamma_2}{2}\varphi_{2\mathbf{k}\lambda} = -(\omega_0 - \omega_{\mathbf{k}})\varphi_{1\mathbf{k}\lambda},$$

where $\tau_0^{-2} = 2\pi c^3 W/\omega_0^2 V$; W is the probability of spontaneous emission of an atom. The term containing N_2 describes the process of spontaneous decay. Let us note that, for example, in work ([2]) this term was obtained incorrectly, since there a system of equations was derived for the mean values of the operators $a_{\mathbf{k}\lambda}$, $a^+_{\mathbf{k}\lambda}$, and not for their bilinear combinations. The physical meaning of the parameter γ_2 is most simply clarified if one considers the beginning of the radiation process in an ideal resonator, when at the time $t = 0$ the number of atoms in the upper level is N_2^0. In this case, from equations (2) it is not difficult to obtain the relation, valid for $\gamma_2 t \gg 1$:

$$\dot{n}_{\mathbf{k}\lambda} \simeq \frac{\gamma_2}{2}\frac{N_2^0}{2\tau_0^2}\frac{1}{\gamma_2^2/4 + (\omega_0 - \omega_{\mathbf{k}})^2}. \tag{3}$$

It is seen from (3) that the spectral distribution of the intensity has a Lorentzian form with width γ_2. For the total number of photons we obtain

$$n = \sum_{\mathbf{k}\lambda} n_{\mathbf{k}\lambda} = N_2^0 W t.$$

The same consideration can readily be carried out for a real resonator. For ruby generators the values of the parameters are as follows: $\gamma_1 \sim 10^9$ sec^{-1}, $W_0 \sim 10^3 \div 10^4$ sec^{-1}, $\gamma_2 \sim 10^{11}$ sec^{-1}, $W \sim 3 \cdot 10^2$ sec^{-1}. In the region of such parameter values the inequalities

$$\dot{\varphi}_{1\mathbf{k}\lambda} \ll \frac{\gamma_2}{2}\varphi_{1\mathbf{k}\lambda}, \qquad \dot{\varphi}_{2\mathbf{k}\lambda} \ll \frac{\gamma_2}{2}\varphi_{2\mathbf{k}\lambda},$$

are valid, since γ_2

significantly exceeds the characteristic frequencies of oscillation of the sought functions in equations (2). In real resonators the ruby sample is usually placed between the mirrors along the optical axis of the resonator. Under such conditions photons multiply only within a small solid angle Ω, which depends on the geometry of the system. Taking this into account, and also assuming the volume of the resonator to be sufficiently large and replacing the sum by an integral, we reduce equations (2) to the form

$$\dot{\eta}(\omega,t)+\gamma_1\eta(\omega,t) = \alpha N_0[2\theta(t)-1]\,\frac{\omega_0\eta(\omega,t)\gamma_2^2}{4\omega\left[\gamma_2^2/4+(\omega_0-\omega)^2\right]}+\frac{\Omega\omega_0\gamma_2\theta(t)}{8\pi\omega\tau_0^2\left[\gamma_2^2/4+(\omega_0-\omega)^2\right]},$$

$$\dot{\theta}(t) - W_0[1-\theta(t)] + \frac{\theta(t)}{\tau} = (1-2\theta)\alpha N_0 \int \frac{4\pi V}{(2\pi c)^3}\,\frac{\omega\omega_0\eta(\omega,t)\gamma_2^2\,d\omega}{4\left[\gamma_2^2/4+(\omega-\omega_0)^2\right]}, \quad (4)$$

where

$$\theta = N_2N_0^{-1}, \qquad \eta(\omega,t) = n(\omega,t)N_0^{-1}, \qquad \alpha = \gamma_2^{-1}\tau_0^{-2}, \qquad \tau^{-1} = W.$$

An investigation of these equations shows that they describe damped oscillations of the light intensity (with a damping decrement, for the above-indicated values of the parameter, of $\sim 10^3 \div 10^4$ s^{-1}) and a stationary regime, which is determined by the conditions $\dot{\eta} = \dot{\theta} = 0$. Let us find the spectral distribution in the stationary regime. From the first equation one can obtain the following expression for η_{stat}:

$$\eta_{\text{stat}} = \frac{\theta_{\text{stat}}\Omega}{2\pi\gamma_1\gamma_2\tau_0^2(\xi^2+a_0)}, \quad (5)$$

where $\xi = 2(\omega-\omega_0)\gamma_2^{-1}$, $a_0 = 1-B_0^{-1}$, $B_0 = \gamma z_0^{-1}$, $z_0 = 2\theta_{\text{stat}}-1$, $\gamma = \gamma_1(\alpha N_0)^{-1}$. Substituting (5) into the second equation (4), we find z_0:

$$z_0 = \gamma+\varepsilon_1^2 z_{10}+\ldots, \quad \text{where } \varepsilon_1 = \frac{\Omega}{4\pi\tau} \ll 1, \quad Z_{10} = -\frac{(\gamma+1)^2\gamma}{4\eta_0^2\gamma_1^2}, \quad \eta_0 = \frac{1+\gamma}{2\gamma_1\tau}\left(\frac{W_0}{W_{\text{p}}}-1\right),$$

$$W_{\text{p}} = \frac{(1+\gamma)}{\tau(1-\gamma)}$$

is the threshold value of the pumping. The expansion of z_0 in a series is valid under the condition $(W_0/W_{\text{p}}-1) \gg \Omega/4\pi$, which is usually well satisfied. Thus, η_{stat} has the form

$$\eta_{\text{stat}}(\omega) = \frac{A_0}{(\omega - \omega_0)^2 + \gamma_0^2/4}, \tag{6}$$

where

$$A_0 = \frac{(1+\gamma)\Omega\gamma_2}{16\pi\gamma_1\tau_0^2}, \qquad \gamma_0 = \frac{\gamma_2\Omega}{4\pi\left(W_0/W_\text{p} - 1\right)}. \tag{7}$$

The quantity γ_0 is the width of the emission spectrum in the stationary regime. Consequently, the spectral width decreases as the solid angle decreases and as the pumping power increases. For example, in resonators with spherical mirrors $\Omega \approx 10^{-3}$ rad. Then, for $W_0 \approx 2W_\text{p}$, we have $\gamma_0 \sim 10^7$ s^{-1}.

Thus, upon reaching the stationary regime the photon spectrum narrows considerably. This narrowing is due to the nonlinear character of the development of the photon avalanche, as a result of which photons whose frequencies are close to the frequency of the line center ω_0 multiply relatively more rapidly.

In conclusion, we note that the description considered in the present work—

radiation by traveling waves assumes, as already mentioned, that the dimensions of the resonator and of the active sample are sufficiently large.

Next, let us find the total number of photons in the resonator:

$$\eta_{\text{stat}} = \frac{Q_{\text{stat}}\Omega}{2\pi^2\tau\gamma_1}\int_0^\infty \frac{d\xi}{\xi^2 + \gamma_0^2\gamma_2^{-2}} = \frac{(1+\gamma)}{2\gamma_1\tau}\left(\frac{W_0}{W_\text{p}} - 1\right) \equiv \eta_0. \tag{8}$$

This expression coincides with the value η_{stat} obtained by other authors ([4,5]) with the aid of kinetic equations. Calculations show that the agreement also holds in the region of the transient regime.

It should be noted, however, that at pump values close to threshold, undamped oscillations of the radiation intensity are observed; these are not described by equations of the type (4). To describe this phenomenon, as it seems to us, a more rigorous accounting of the geometrical properties of the resonator and of the active medium is necessary.

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Note: Figure translations are in progress. See original paper for figures.