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Abstract

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On the Theory of the First Variation

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This note sets forth a theory of the first variation for variational problems of a very general kind (including problems of optimal control). We make essential use of the considerations developed in (^{1}).

We shall consider families

$$F = \{f(x,t)\} \tag{1}$$

of n-dimensional vector functions $f(x,t)$, defined for $x \in G,\ t \in I$, where G is a domain of n-dimensional space R^n, and I is an interval of the time axis. It is assumed that each function $f(x,t)$ of the family is measurable in t for fixed x, of class C^1 in x for fixed t, and for each compact set $X \subset G$ is majorized in modulus by some function $m(t)$ integrable on I (depending on the choice of f, X):

$$|f(x,t)| \leq m(t), \quad x \in X, \quad t \in I.$$

We shall call the family (1) **quasiconvex** if, for any compact set $X \subset G$, any prescribed nonnegative numbers α, β satisfying $\alpha + \beta = 1$, and arbitrarily small $\varepsilon > 0$, together with functions $f_1(x,t), f_2(x,t)$ in F, the family F also contains a function

$$f(x,t) = \alpha f_1(x,t) + \beta f_2(x,t) + g(x,t),$$

where $g(x,t)$ satisfies, for $x \in X$ and any $t_1, t_2 \in I$, the inequality

$$\left| \int_{t_1}^{t_2} g(x,t)\, dt \right| \leq \varepsilon.$$

The following examples of quasiconvex families F_∞, F_p are most often encountered in variational problems.

Let $f(x,u,t)$ be an n-dimensional function defined for $x \in G,\ t \in I,\ u \in R^r$, of class C^1 in x for fixed u,t, and measurable jointly in u,t for fixed x; moreover, suppose that for any compact sets $X \subset G,\ Y \subset R^r$ there exists a function $m(t)$

integrable on I (depending on the choice of X, Y) which majorizes $|f(x,u,t)|$ for $x \in X,\ u \in Y,\ t \in I$:

$$|f(x,u,t)| \leq m(t).$$

By U_t denote an arbitrary family of subsets of the space R^r, depending on $t \in I$.

The family F_∞ consists of functions of the form $f(x,u(t),t)$, where $u(t)$ is an arbitrary measurable function, essentially bounded on I, satisfying almost everywhere on I the condition $u(t) \in U_t$.

Let $f(x,u,t)$ satisfy the additional requirement: if $u(t)$ is a function measurable on I and

$$\int_I |u(t)|^p \, dt < \infty,$$

where the number $p \geq 1$ is given, and X is an arbitrary compact set in G, then there exists a function $m(t)$ integrable on I (depending on $u(t), X$) such that, for $x \in X,\ t \in I$,

$$|f(x,u(t),t)| \leq m(t).$$

The family F_p consists of functions of the form $f(x,u(t),t)$, where $u(t)$ is an arbitrary function measurable on I satisfying the inequality

$$\left(\int_I |u(t)|^p \, dt \right)^{1/p} \leq 1.$$

The quasiconvexity of the families F_∞, F_p follows directly from the following approximation lemma (see (^{1})).

Lemma. Let $f_i(x,t)$, $x \in X \subset G$, X a compact set, $t \in I$, $i = 1, \ldots, s$, be n-dimensional vector functions with the structural properties listed in the definition of the family F. Let, further, $p_1(t), \ldots, p_s(t)$ be nonnegative measurable functions satisfying almost everywhere on I the equality

$$\sum_{i=1}^{s} p_i(t) = 1.$$

Then for any $\varepsilon > 0$ the interval I can be subdivided into sufficiently small subintervals E_j, $j = 1, 2, \ldots$, and to each E_j one can assign, in this way, one of the functions $f_i(x,t)$ (denote it by $f_{E_j}(x,t)$), such that the function $f(x,t)$, defined by the equality $f(x,t) = f_{E_j}(x,t)$ for $t \in E_j$, $j = 1, 2, \ldots$, satisfies the relation

$$\left| \int_{t_1}^{t_2} \left(\sum_{i=1}^{s} p_i(t) f_i(x,t) - f(x,t) \right) dt \right| \leq \varepsilon$$

for $x \in X$, $t_1, t_2 \in I$.

Consequently, the function $f(x, t)$ is of class C^1 with respect to x for fixed t, is measurable with respect to t for fixed x, and is majorized by some summable function $m(t)$: $|f(x, t)| \leq m(t)$ for $x \in X$, $t \in I$.

Now let, in the $2 + 2n = m$-dimensional space of the variables τ_1, τ_2, x_1, x_2, where τ_1, τ_2 are scalars from I, and x_1, x_2 are vectors from G, there be given an $(m-k+1)$-dimensional manifold N with $(m-k)$-dimensional boundary M. Denote by $N_T(\tau_1, \tau_2, x_1, x_2)$ the $(m-k+1)$-dimensional tangent half-plane tangent to N at the point $(\tau_1, \tau_2, x_1, x_2) \in M$.

Consider the differential equation

$$\dot{x} = f(x, t), \tag{2}$$

where f is some function from the quasiconvex family F, and let

$$z(t), \qquad t_1 \leq t \leq t_2, \tag{3}$$

be a solution of this equation with boundary values

$$z(t_1) = z_1, \qquad z(t_2) = z_2. \tag{4}$$

We denote the point (t_1, t_2, z_1, z_2) by q_z.

Similarly, if

$$\dot{x} = \tilde{f}(x, t), \qquad \tilde{f} \in F, \tag{5}$$

is any other equation with right-hand side from the quasiconvex family F, and $\tilde{x}(t)$, $\tau_1 \leq t \leq \tau_2$, is some solution of this equation, then by $q_{\tilde{x}}$ we shall denote the $(2+2n)$-dimensional point $(\tau_1, \tau_2, \tilde{x}(\tau_1), \tilde{x}(\tau_2))$.

The set of all possible $q_{\tilde{x}}$, where $\tilde{x}$ is an arbitrary solution of an arbitrary equation of the form (5), will be denoted by Q.

The solution (3) of equation (2) will be called **extremal with respect to the quasiconvex family (1) and the given manifold N of boundary values** if $q_z = (t_1, t_2, z_1, z_2) \in M$ and the point q_z can be surrounded in N by so small a neighborhood that the intersection of the set Q with this neighborhood lies entirely on the boundary M of the manifold N.

To simplify the formulation of the transversality conditions, assume that the function $f(z(t), t)$ is continuous at the points t_1, t_2.

Necessary condition for extremality (maximum principle). Let the solution (3) of equation (2), satisfying the boundary conditions (4), be extremal with respect to the quasiconvex family (1) and the given manifold N of boundary values.

Then there exists a nontrivial solution $\psi(t)$, $t_1 \leq t \leq t_2$, of the equation

$$\dot{\psi} = -\psi f_x(z(t), t), \tag{6}$$

such that the "maximum condition" is satisfied,

$$\int_{t_1}^{t_2} \psi(t) f(z(t), t)\, dt \geq \int_{t_1}^{t_2} \psi(t) \tilde{f}(z(t), t)\, dt \tag{7}$$

for any $\tilde{f}(x, t) \in F$, and the $(2 + 2n)$-vector given by the row

$$(\psi(t_1) f(z(t_1), t_1),\ -\psi(t_2) f(z(t_2), t_2),\ -\psi(t_1),\ \psi(t_2)), \tag{8}$$

is orthogonal to M at the point (t_1, t_2, z_1, z_2)—the transversality condition—and is directed toward the half-space $N_T(t_1, t_2, z_1, z_2)$.

The necessary condition formulated here is equivalent to the Pontryagin maximum principle (see (2)) for the families F_∞. For the families F_p, the original formulation of the maximum principle is unsuitable, and it should be replaced by the maximum condition (7).

For example, if $f(x, u, t) = A(t)x + B(t)u + f(t)$, $A(t)$, $B(t)$, $f(t)$ are measurable, and $|A(t)|$, $|f(t)|$, $|B(t)|^q$, where $q = p/(p-1)$, are integrable on I, then the maximum condition (7) gives

$$\int_{t_1}^{t_2} \psi(t) B(t) u(t)\, dt \geq \int_{t_1}^{t_2} \psi(t) B(t) \tilde{u}(t)\, dt,$$

if $\|u(t)\|_p \leq 1$, $\|\tilde{u}(t)\|_p \leq 1$

$$\left(\|u\|_p = \left(\int_{t_1}^{t_2} |u|^p\, dt \right)^{1/p} \right).$$

Consequently,

$$u(t) = \frac{\psi(t) B(t) |\psi(t) B(t)|^{q-2}}{\left(\|\psi(t) B(t)\|_q\right)^{q-1}}$$

when $\psi(t) B(t) \neq 0$, and $u(t) = 0$ when $\psi(t) B(t) = 0$.

Proof of the necessary condition. For any function $\tilde{f}$ from the quasiconvex family (1) and any $1 > \varepsilon > 0$, there exists a function $g(x,t)$ such that $f + \varepsilon(\tilde{f} - f) + g = f + \varepsilon\delta f + g \in F$, and, for any x from some neighborhood of $z(t)$, $t_1 \leq t \leq t_2$, and $t \in I$,

$$\left| \int_{t_1}^{t_2} g(x,t)\, dt \right| \leq \varepsilon^2. \tag{9}$$

Consequently, for sufficiently small ε the equation

$$\dot{x} = f(x,t) + \varepsilon\delta f(x,t) + g(x,t)$$

has a solution $x(t)$ on the interval $t_1 + \varepsilon\delta t_1 \leq t \leq t_2 + \varepsilon\delta t_2$, satisfying the initial value $x(T) = z(T) + \varepsilon\delta x$, where the prescribed numbers δt_1, δt_2 and the vector δx are arbitrary, and T is a point of the interval $t_1 < t < t_2$ fixed throughout the proof.

Let $\Phi(t)$ denote the fundamental matrix of solutions of the equation $\delta\dot{z} = f_x(z(t),t)\delta z$. Estimate (9) gives

$$x(t) = z(t) + \varepsilon\Phi(t)\left(\Phi^{-1}(T)\delta x + \int_T^t \Phi^{-1}(\tau)\delta f(z(\tau),\tau)\, d\tau\right) + o(\varepsilon) =$$

$$= z(t) + \varepsilon\delta z(t) + o(\varepsilon). \tag{10}$$

Further,

$$x(t_i + \varepsilon\delta t_i) = x_i = z(t_i) + \varepsilon\big(f(z(t_i),t_i)\delta t_i + \delta z(t_i)\big) + o(\varepsilon) =$$

$$= z(t_i) + \varepsilon\delta x_i + o(\varepsilon), \qquad i = 1,\ 2. \tag{11}$$

The set of all possible $(2+2n)$-vectors of the form $(\delta t_1,\ \delta t_2,\ \delta x_1,\ \delta x_2)$, laid off from the point $(t_1,\ t_2,\ z_1,\ z_2)$, forms a convex cone K in R^{2+2n} with vertex at $(t_1,\ t_2,\ z_1,\ z_2)$. The cone K and the half-plane $N_T(t_1,t_2,z_1,z_2)$ are separable; otherwise, for sufficiently small $\varepsilon > 0$ there would exist points

$$(t_1 + \varepsilon\delta t_1,\ t_2 + \varepsilon\delta t_2,\ x(t_1 + \varepsilon\delta t_1),\ x(t_2 + \varepsilon\delta t_2)),$$

belonging to N, but not lying on the boundary M, which would contradict the assumption that $z(t)$ is extremal.

Consequently, there exists a $(2+2n)$-row $(\chi_1, \chi_2, \rho_1, \rho_2)$, where χ_1, χ_2 are scalars and ρ_1, ρ_2 are n-rows, orthogonal to M at the point (t_1, t_2, z_1, z_2), directed toward the half-plane $N_T(t_1, t_2, z_1, z_2)$, and satisfying the condition

$$\chi_1 \delta t_1 + \chi_2 \delta t_2 + \rho_1 \delta x_1 + \rho_2 \delta x_2 \le 0, \qquad (\delta t_1, \delta t_2, \delta x_1, \delta x_2) \in K. \tag{12}$$

We shall show that (8) may be taken as such a row. From (10)–(12) we obtain the condition

$$\sum_{i=1}^{2} \left[(\chi_i + \rho_i f(z(t_i), t_i))\delta t_i + \rho_i \Phi(t_i)\Phi^{-1}(T_i)\delta x + \rho_i \Phi(t_i) \int_T^{t_i} \Phi^{-1} \delta f \, dt \right] \le 0. \tag{13}$$

Since $\delta t_1, \delta t_2, \delta x$ are arbitrary, it follows that

$$\chi_i + \rho_i f(z(t_i), t_i) = 0, \qquad \sum_{i=1}^{2} \rho_i \Phi(t_i) = 0. \tag{14}$$

Therefore inequality (13) becomes

$$\rho_2 \Phi(t_2) \int_{t_1}^{t_2} \Phi^{-1} \delta f \, dt \le 0,$$

which coincides with (7), if we introduce the notation

$$\psi(t) = \rho_2 \Phi(t_2) \Phi^{-1}(t).$$

Consequently, the equalities (14) give

$$\psi(t_1) = -\rho_1, \qquad \psi(t_2) = \rho_2, \qquad \chi_1 = \psi(t_1) f(z(t_1), t_1), \qquad \chi_2 = -\psi(t_2) f(z(t_2), t_2).$$

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Note: Figure translations are in progress. See original paper for figures.

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