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Abstract

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On the Rate of Convergence to the Normal Distribution

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1. Let

$$\xi_1,\ \xi_2, \dots, \xi_n, \dots$$

be a sequence of independent identically distributed random variables with common distribution function (d.f.) $F(x)$ and characteristic function (c.f.) $f(t)$. We shall assume that the d.f. $F(x)$ belongs to the domain of attraction of the normal law

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2}\, dz.$$

In this case the variables ξ_j have a finite first moment (([1], p. 192), and we shall assume that $E\xi_j = 0$. By $F_n(x)$ we denote the d.f. of the sum

$$S_n = \frac{\xi_1 + \dots + \xi_n}{B_n} - A_n,$$

normalized so that $F_n(x) \to \Phi(x)$.

There are simple sufficient conditions guaranteeing that

$$\sup_x |F_n(x) - \Phi(x)| = O(n^{-\delta/2}), \tag{1}$$

where $0 < \delta \leq 1$ (([1], p. 215).

In Theorem 1 of the present note we give necessary and sufficient conditions for relation (1) to hold.

2. Theorem 1. *In order that*

$$|F_n(x) - \Phi(x)| = O(n^{-\delta/2}), \qquad 0 < \delta \leq 1,$$

it is necessary and sufficient that the following conditions hold:

$$1) \quad \sigma^2 = \int x^2 \, dF(x) < \infty;$$

$$2) \quad \int_{|x|>z} x^2 \, dF(x) = O(z^{-\delta}), \qquad z \to \infty.$$

In this case the normalizing constants necessarily have the form $A_n = 0$, $B_n = \sigma\sqrt{n}$.

Remark. From condition 2) it follows that all moments of order less than $2+\delta$ are finite.

It is easy to see that the d.f. $F_n(x)$ for lattice distributions $F(x)$ cannot be approximated by the continuous function $\Phi(x)$ with an accuracy exceeding $1/\sqrt{n}$. Therefore, in wishing to extend the results of Theorem 1 to exponents $\delta > 1$, we must impose some additional requirement on the d.f. $F(x)$. As such a requirement we choose Cramér' s condition (C).

We shall say that the d.f. $F(x)$ satisfies condition (C) if

$$\overline{\lim}_{t\to\infty} |f(t)| < 1. \tag{C}$$

Theorem 2. In order that

$$|F_n(x) - \Phi(x)| = O(n^{-\delta/2}), \qquad 2k < \delta \le 2(k+1), \qquad k = 0, 1, \dots,$$

it is necessary, and for distributions satisfying condition (C) sufficient, that the following conditions hold:

1) all integral moments of the d.f. $F(x)$ of order less than $\delta+2$ are finite and coincide with the corresponding moments of the normal distribution;

2)

$$\int_{|x|>z} x^{2(k+1)} \, dF(x) = O(z^{2k-\delta}), \qquad \text{if } \delta < 2(k+1),$$

$$\int x^{2(k+2)} \, dF(x) < \infty, \qquad \text{if } \delta = 2(k+1).$$

Corollary. If for all $\delta > 0$

$$|F_n(x) - \Phi(x)| = O(n^{-\delta/2}), \tag{2}$$

then the d.f. $F(x)$ is normal.

Remark. For d.f.' s $F(x)$ belonging to the domains of attraction of stable laws with exponent $\alpha < 2$, a superfast convergence to the limiting distribution similar to (2) is quite possible. Whatever stable d.f. $G_\alpha(x)$ with exponent $\alpha < 2$ is given, there exists a d.f. $F(x)$, not stable, for which

$$|F_n(x) - G_\alpha(x)| = O(e^{-cn}), \qquad c > 0.$$

3. We now give analogues of Theorem 1 corresponding to local limit theorems ((1), pp. 238-258). Let first the random variables ξ_j be lattice-valued with span h. Without loss of generality one may suppose that the possible values of ξ_j are lh, $l = 0, \pm 1, \ldots$.

It follows from Theorem 1 that one may restrict oneself to the case $\mathbf{E}\xi_j = 0$, $\mathbf{D}\xi_j = 1$. Then $A_n = 0$, $B_n = \sqrt{n}$. Put

$$P_n(l) = \mathbf{P}\{\sqrt{n} S_n = lh\}, \qquad l = 0, \pm 1, \ldots$$

By $\varphi(x) = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$ we denote the normal density.

Theorem 3. In order that

$$\sup_{-\infty<l<\infty} \left| \frac{\sqrt{n}}{h} P_n(l) - \varphi\left(\frac{lh}{\sqrt{n}}\right) \right| = O(n^{-\delta/2}),$$

it is necessary and sufficient that:

1) the d.f. $F(x)$ satisfy conditions 1), 2) of Theorem 2;

2) the span of the distribution h be maximal.

Put $p_n(x) = F_n'(x)$.

Theorem 4. In order that

$$\sup_{-\infty<x<\infty} |p_n(x) - \varphi(x)| = O(n^{-\delta/2}), \qquad \delta > 0,$$

it is necessary and sufficient that:

1) the d.f. $F(x)$ satisfy conditions 1), 2) of Theorem 2;

2) there exist a number N such that the d.f. $F_N(x)$ is absolutely continuous and

$$\sup_x p_N(x) < \infty.$$

4. The proofs of Theorems 1–4 are based on similar ideas. We briefly outline the proof of Theorem 1, restricting ourselves to symmetric distributions $F(x)$ and assuming, for simplicity, that already $\sigma^2 < \infty$. Then one may take $\sigma = 1$, $B_n = \sqrt{n}$.

In a neighborhood of zero the characteristic function is $f(t) = \exp\{-\frac{1}{2}t^2(1 + \gamma(t))\}$, where $\gamma(t) = o(1)$, $t \to 0$.

Lemma. *Under the assumptions made above, equality (1) holds if and only if*

$$\int_0^x t^2|\gamma(t)|\,dt = O(x^{3+\delta}), \qquad x \to 0. \tag{3}$$

Proof. Necessity. In a neighborhood of zero the difference $f(t) - e^{-t^2/2}$ can vanish only a finite number of times (if $f(t) \not\equiv e^{-t^2/2}$), and consequently, in some neighborhood of zero the function $\gamma(t)$ has constant sign. Therefore, using Parseval' s equality, we have

$$\frac{1}{\sqrt{2\pi}}\left|\int_{-\infty}^{\infty}[F_n(x) - \Phi(x)]\,xe^{-x^2/2}\,dx\right| = \left|\int_{-\infty}^{\infty}[f_n(t) - e^{-t^2/2}]e^{-t^2/2}\,dt\right| \geq$$

$$\geq \frac{1}{e}\int_0^1 \left|1 - e^{\frac{1}{2}t^2\gamma(t/\sqrt{n})}\right|\,dt + O(e^{-\ln^2 n}),$$

$$\int_0^{1/\sqrt{n}} t^2|\gamma(t)|\,dt = O\big(n^{-(3+\delta)/2}\big)\,,$$

which is equivalent to (3).

The proof of the sufficiency of (3) is obtained by the usual methods connected with the application of Esseen' s theorem ((1), pp. 211–218).

Subsequently the proof of the theorem is continued as follows. It is proved that (3) is equivalent to the equality

$$\int \left(\frac{\sin xu}{xu} - 1 + \frac{x^2u^2}{6}\right)dF(u) = O(x^{2+\delta}), \qquad x \to 0,$$

which, as is not difficult to verify, is already equivalent to condition 2) of Theorem 1.

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REFERENCES

1. B. V. Gnedenko, A. I. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, Moscow—Leningrad, 1949.

Note: Figure translations are in progress. See original paper for figures.

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