ON THE THEORY OF GENERALIZED ALMOST-PERIODIC SEQUENCES

MATHEMATICS

1965

SovietRxiv

Abstract

Full Text

UDC 517.566.5

MATHEMATICS

I. S. FROLOV

ON THE THEORY OF GENERALIZED ALMOST-PERIODIC SEQUENCES

(Presented by Academician I. G. Petrovskii, March 26, 1965)

In the present note we study sequences that are almost periodic with respect to generalized shift operators generated by ultraspherical polynomials. The main result of the note consists in obtaining Parseval' s equality for such sequences.

Let $P_n^\nu(x)$, $n = 0, 1, 2, \dots$; $\nu > 1$, be polynomials orthogonal on the interval $[-1, 1]$ with weight $d\Omega = (1 - x^2)^{\nu - 1/2} dx$, normalized by the condition $P_n^\nu(1) = 1$.

Then (1)

$$\int_{-1}^{1} P_n^\nu(x) P_m^\nu(x)\, d\Omega = \frac{\delta_m^n}{\omega_n^\nu},$$

where

$$\omega_n^\nu = \frac{\nu + n}{\nu} \frac{\Gamma(n + 2\nu)}{\Gamma(2\nu) n!} = \mathcal{O}(n^{2\nu}).$$

Let

$$P_k^\nu(x) P_j^\nu(x) = \sum_{n=0}^{\infty} C_{kjn} P_n^\nu(x) \omega_n^\nu.$$

The quantities $C_{kjn}\omega_n^\nu$ can be written explicitly (2); in particular, $C_{kjn} \neq 0$ if $k + j + n$ is even and $\max(k, j, n) \leq \sigma$, where $2\sigma = k + j + n$; otherwise $C_{kjn} = 0$.

We shall consider sequences x_r such that $|x_r| \ll a / \sqrt{\omega_r^{\nu - 1}}$, where a is a constant.

Definition of an almost-periodic (a.p.) sequence. A sequence x_r is called **almost periodic with respect to the matrix** C_{kjn} if the family of generalized shifts

$$\sum_{r=0}^{\infty}\sqrt{\omega_p^\nu\omega_q^\nu C_{pqr}\omega_r^\nu x_r}$$

is compact in the sense of uniform convergence.

The basic properties of a.p. sequences were studied by B. M. Levitan in (^{3,4}). For a.p. sequences with respect to generalized shift operators (g.s.o.) generated by arbitrary orthogonal polynomials, he proved Parseval' s equality only for a folded sequence. In the present note it is shown that, in the case of ultraspherical polynomials, Parseval' s equality holds already for the a.p. sequence itself.

For sequences x_r, y_r, define the scalar product

$$\{x_r, y_r\} = \lim_{N_k\to\infty}\frac{1}{N_k}\sum_{r=0}^{N_k-1}\omega_r^\nu x_r y_r.$$

Introduce the Fourier coefficients for x_r:

$$A_j = \lim_{N_k\to\infty}\frac{1}{N_k}\sum_{r=0}^{N_k-1}x_r\frac{P_r^\nu(x_j)}{\alpha_j}\omega_r^\nu,$$

where

$$\alpha_j^2 = \lim_{N_k\to\infty}\frac{1}{N_k}\sum_{r=0}^{N_k-1}\left(P_r^\nu(x_j)\right)^2\omega_r^\nu$$

(N_k is some unboundedly increasing sequence of integers) (see (3)).

Theorem. *If x_r is an a.p. sequence, then Parseval' s equality holds*

$$\{x_r, x_r\} = \sum_j A_j^2.$$

The proof of this theorem is based on a lemma, which will be proved below.

Let β_r^p be the transition matrix from the polynomials $P_p^\nu(x)$ to the Chebyshev polynomials $T_p(x)$:

$$T_r(x) = \sum_p \beta_r^p P_p^\nu(x)$$

(the summation is finite), and let

$$T_{pT}q = \sum_{r=0}^{\infty} a_{pqr} T_r.$$

Then (4)

$$a_{pqr} = \sum_{k,j,n} \beta_p^k \beta_q^j C_{kjn} \omega_n^\nu \alpha_r^n,$$

where α_r^n is the matrix inverse to β_n^r:

$$\sum_r \alpha_r^n \beta_m^r = \delta_m^n.$$

The order of summation over k, j, n, in view of the finiteness of the summation for fixed p, q, r, is immaterial.

For ultraspherical polynomials the asymptotic formula holds ((5) in other notation)

$$P_r^\nu = c \sin^{-\nu}\theta \sqrt{\omega_r^{\nu^{-1}}} \left\{ \cos[(r+\nu)\theta + \gamma] + (r\sin\theta)^{-1} O(1) \right\}; \tag{1}$$

c is a certain constant; $\gamma = \pi/2 - \nu$.

The estimate of the remainder term is uniform on $[c_1 r^{-1}, \pi - c_1 r^{-1}]$, c_1 a constant.

Lemma. *If* $|x_p| < a\sqrt{\omega_p^{\nu^{-1}}}$, *then* $\sum_p |\beta_r^p|\,|x_p| < A$ *for* $\nu > 1$, ν *nonintegral.*

Proof. Using the orthogonality of $P_r^\nu(x)$, $P_s^\nu(x)$ on $[-1, 1]$, we have

$$\beta_r^p = \int_0^\pi P_p^\nu(\cos\theta) T_r \omega_p^\nu \, d\Omega = \int_0^\pi P_p^\nu(\cos\theta)\cos(r\theta)\, \omega_p^\nu \sin^{2\nu}\theta \, d\theta;$$

$$\beta_r^p \sqrt{\omega_p^{\nu^{-1}}} = \int_0^\pi P_p^\nu(\cos\theta)\cos(r\theta) \sqrt{\omega_p^\nu} \sin^{2\nu}\theta \, d\theta.$$

We shall use the asymptotic formula (1) and represent $\beta_r^p \sqrt{\omega_p^{\nu^{-1}}}$ in the form

$$\int_0^{p^{-1}} P_p^\nu(\cos\theta)\cos(r\theta)\sqrt{\omega_p^\nu}\sin^{2\nu}\theta\,d\theta + \int_{\pi-p^{-1}}^{\pi} P_p^\nu(\cos\theta)\cos(r\theta)\sqrt{\omega_p^\nu}\sin^{2\nu}\theta\,d\theta$$

$$+\int_{p^{-1}}^{\pi-p^{-1}} C\cos[(p+\nu)+\gamma]\cos(r\theta)\sin^{\nu}\theta\,d\theta+\int_{p^{-1}}^{\pi-p^{-1}} C_1(p\sin\theta)^{-1}\cos(r\theta)\sin^{\nu}\theta\,d\theta.$$

Using the fact that the family $P_p\sqrt{\omega_p^{\nu}}\sin^{\nu}\theta$ is bounded on $[0,\pi]$ uniformly in p ((2), p. 296), it is easy to estimate the first two integrals:

$$\left|\int_0^{p^{-1}} D_p^{\nu}\cos(r\theta)\sqrt{\omega_p^{\nu}\sin^2\theta}\,d\theta\right|\le k\int_0^{p^{-1}}\sin^{\nu}\theta\,d\theta\le$$

$$\le k\int_0^{p^{-1}}\theta^{\nu}\,d\theta=\frac{k}{\nu+1}\theta^{\nu+1}\Big|_0^{p^{-1}}=\frac{k}{\nu+1}\left(\frac{1}{p}\right)^{\nu+1};\qquad\left|\int_{\pi-p^{-1}}^{\pi}\right|\le\frac{k}{\nu+1}\left(\frac{1}{p}\right)^{\nu+1}.$$

The product of cosines in the third integral is transformed into a sum, and the resulting expression is integrated by parts. Then each of the terms is estimated:

$$1)\ \left|\frac{-1/2}{r-p-\nu}\int_{p^{-1}}^{\pi-p^{-1}}\sin\left[(r-p-\nu)\theta+\gamma\right]\sin^{\nu-1}\theta\cos\theta\,d\theta\right|\le\frac{\alpha}{(r-p-\nu)^2},$$

α does not depend on r;

$$2)\ \left|\frac{1/2}{r-p-\nu}\sin\left[(r-p-\nu)\theta+\gamma\right]\sin^{\nu}\theta\Big|_{p^{-1}}^{\pi-p^{-1}}\right|\le\frac{1/2}{|r-p-\nu|}\left(\frac{1}{p}\right)^{\nu};$$

$$3)\ \left|\frac{1/2}{r+p+\nu}\int_{p^{-1}}^{\pi-p^{-1}}\sin\left[(r+p+\nu)\theta+\gamma\right]\sin^{\nu-1}\theta\cos\theta\,d\theta\right|\le\frac{\beta}{(r+p+\nu)^2},$$

β does not depend on r;

$$4)\ \left|\frac{-1/2}{r+p+\nu}\sin\left[(r+p+\nu)\theta+\gamma\right]\sin^{\nu}\theta\Big|_{p^{-1}}^{\pi-p^{-1}}\right|\le\frac{1/2}{|r+p+\nu|}\left(\frac{1}{p}\right)^{\nu}.$$

Let us estimate the last integral:

$$\left|\int_{p^{-1}}^{\pi-p^{-1}} c_1p^{-1}\sin^{\nu-1}\theta\cos(r\theta)\,d\theta\right|\le\frac{\delta}{pr};$$

δ does not depend on r; $\beta_r^p = 0$ for $p > r$; upon summation over p, all sums are uniformly bounded with respect to r, whence the assertion of the lemma follows.

In proving the theorem we shall regard the following facts as known (see (3)):

1) For the a.p. x_r, the convolution z_p

$$z_p = \lim_{N_k \to \infty} \frac{1}{N_k} \sum_{q=0}^{N_k - 1} \omega_q^\nu x_q \sum_{r=0}^{\infty} C_{pqr} \omega_r^\nu x_r$$

is almost periodic.

2) For z_p the following integral representation holds:

$$z_p = \int_{-1}^{1} P_p^\nu(x)\, d\sigma(x) = \int_{-1}^{1} P_p^\nu(x)\, dD(x) + \int_{-1}^{1} P_p^\nu(x)\, dS(x) = z_p' + z_p'';$$

$\sigma(x)$ is a monotone bounded function, $D(x)$ is the jump function, $S(x)$ is the continuous part.

3) z_p'' is a.p.; $\{z_p', z_p''\} = 0$.

4) The Fourier coefficients of the convolution z_p are equal to

$$\left\{ z_p, \frac{G_p^\nu(x_j)}{\alpha_j} \right\} = \alpha_{jA} j^2;$$

A_j are the Fourier coefficients for x_p, A_j^2 is the jump of the function $D(x)$ at the point x_j.

Proof of the theorem.

$$z_0 = \lim_{N_k \to \infty} \frac{1}{N_k} \sum_{q=0}^{N_k - 1} \omega_q^\nu x_q \sum_{r=0}^{\infty} C_{0qr} \omega_r^\nu x_r = \lim_{N_k \to \infty} \frac{1}{N_k} \sum_{q=0}^{N_k - 1} \omega_q^\nu x_q^2 = \{x_q, x_q\},$$

but

$$z_0 = z_0' + z_0'' = \int_{-1}^{1} P_0^\nu(x)\, dD(x) + \int_{-1}^{1} P_0^\nu(x)\, dS(x) = \sum_j A_j^2 + \int_{-1}^{1} dS(x).$$

It remains to prove that $S(x) = \text{const}$.

From the almost-periodicity of z''_p there follows, by (3), the existence of a finite ε-net in the family

$$\sum_n \sqrt{\omega_j^\nu \omega_k^\nu}\, C_{jkn}\omega_n^\nu z''_n$$

with respect to j, uniformly in k, i.e.

$$\left|\sum_n \sqrt{\omega_{j'}^\nu}\, C_{j'kn}\omega_n^\nu z''_n - \sum_n \sqrt{\omega_{j''}^\nu}\, C_{j''kn}\omega_n^\nu z''_n\right| \leqslant \varepsilon\sqrt{\omega_k^{-1}};$$

$j' = j_1, j_2, \dots, j_m$; j'' arbitrary.

Multiplying both sides of this inequality by β_p^k and summing over k, we obtain:

$$\left|\sum_{n,k} \sqrt{\omega_{j'}^\nu}\, \beta_p^k C_{j'kn}\omega_n^\nu z''_n - \sum_{n,k} \sqrt{\omega_{j''}^\nu}\, \beta_p^k C_{j''kn}\omega_n^\nu z''_n\right| \leqslant l\varepsilon;$$

the constant l is determined by the lemma.

Thus, for the family

$$\sum_{n,k} \sqrt{\omega_j^\nu}\, \beta_p^k C_{jkn}\omega_n^\nu z''_n$$

(p is a parameter) there exists a finite $l\varepsilon$-net; consequently, by (3), and conversely, one can indicate a finite $l\varepsilon$-net with respect to p (j is a parameter):

$$\left|\sum_{n,k} \beta_{p'}^k C_{jkn}\omega_n^\nu z''_n - \sum_{n,k} \beta_{p''}^k C_{jkn}\omega_n^\nu z''_n\right| \leqslant \varepsilon l\sqrt{\omega_j}.$$

Analogously to the preceding, in the family

$$\sum_{k,j,n} \beta_p^k \beta_q^j C_{jkn}\omega_n^\nu z''_n$$

there is, for every ε, a finite εl^2-net.

For z''_s consider the transformation $\tilde{z}''_r = \sum_s \beta_r^s z''_s$. It is obvious that

$$\sum_{k,j,n} \beta_p^k \beta_q^j C_{jkn}\omega_n^\nu z''_n = \sum_{k,j,n,r,s} \beta_p^k \beta_q^j C_{jkn}\alpha_n^r \omega_n^\nu \beta_r^s z''_s;$$

the order of summation over k, j, n is arbitrary, while over r, s it is first over r, then over s.

But since $\sum_r |\alpha_n^r| = 1$ because of the positivity of α_n^r, it follows that $\sum_{s,r} |\alpha_n^r|\,|\beta_r^s|\,|x_s| < c$; the order of summation over r, s is therefore also immaterial, and instead of the preceding sum one may write

$$\sum_{k,j,n,r} \beta_p^k \beta_q^j C_{jkn} \omega_n^\nu \alpha_n^r \tilde{z}_r'' = \sum_r a_{pqr} \tilde{z}_r''.$$

This family is compact in the sense of uniform convergence. Therefore $\tilde{z}_r''$ is an even a.p. sequence (4),

$$\tilde{z}_r'' = \int_{-1}^{1} \beta_r^s P_s^\nu(x)\, dS = \int_{-1}^{1} T_r(x)\, dS(x), \quad \text{and by (3)} \quad \{\tilde{z}_{r''}, \tilde{z}_{r'}\} = 0,$$

hence $\tilde{z}_r'' \equiv 0,\ s = \text{const}$. The theorem is proved.

I consider it my duty to express gratitude to my adviser B. M. Levitan for posing the problem.

Moscow State University
named after M. V. Lomonosov

Received
16 II 1965

References Cited

1. S. Bochner, Proc. Nat. Acad. Sci. U.S.A., **40**, 1141 (1954).
2. R. Askey, I. Hirshman, Trans. Am. Math. Soc., **91**, No. 2, 294 (1959).
3. B. M. Levitan, Matem. sborn., **16** (58), 3, 259 (1945); **17** (59), 1, 9 (1945).
4. B. M. Levitan, UMN, **4**, 1 (29), 3 (1949).
5. G. Szegő, *Orthogonal Polynomials*, Moscow, 1962, p. 204.