ON THE DECOMPOSABILITY OF ALGEBRAS OF A CERTAIN CLASS INTO A DIRECT PRODUCT OF SIMPLE ALGEBRAS

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Abstract

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ON THE DECOMPOSABILITY OF ALGEBRAS OF A CERTAIN CLASS INTO A DIRECT PRODUCT OF SIMPLE ALGEBRAS

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In the paper ([1]) the following theorems are proved:

1. Every nontrivial normal divisor of a group G is contained in a true direct factor of the group G if and only if G is a direct product (in the narrow sense) of simple groups.

2. A group G is isomorphic to a subdirect product of a family of simple groups if and only if, for every nonunit normal divisor N of the group G, there exists a nontrivial normal divisor N' of the group G such that $NN' = G$.

Here analogous results are given for a broader class of algebras containing, in particular, loops.

Consider a certain class of algebras of the same type with a distinguished unary operation ε (a basic one) such that $\varepsilon(x) = \varepsilon(y) = e$ and $f(e, \dots, e) = e$, where f is an arbitrary basic operation of an algebra of the class under consideration; e is the distinguished element of the algebra; x, y are arbitrary elements of the algebra, and with two, possibly derived, binary operations $\theta(x, y)$ and $\chi(x, y)$ such that: $\theta(x, e) = \theta(e, x) = x$; $\chi(x, e) = x$; $\chi(x, y) = e$ if and only if $x = y$. We shall also require that the class be closed with respect to homomorphisms.

It is easy to see that this class contains the class of loops, as well as the following primitive classes: 1) the class of algebras with distinguished operation ε and binary operations $\theta(x, y)$, $\chi(x, y)$, satisfying the requirements: $\theta(x, e) = \theta(e, x) = x$; $\chi(x, e) = x$; $\chi(x, x) = e$; $\theta(\chi(x, y), y) = x$; 2) the class of algebras with distinguished operation ε and a binary operation χ such that $\chi(x, e) = x$, $\chi(x, \chi(x, y)) = y$ (here the operation θ may be $\theta(x, y) = \chi(x, \chi(e, y))$). It can be shown that the distinguished class of algebras is not equivalent (rationally) to loops.

By a normal divisor of an algebra we shall mean a subalgebra that is the complete inverse image of the distinguished element under a homomorphism of the algebra (the kernel of the homomorphism). A simple algebra is an algebra having no nontrivial normal divisors.

By the direct product (in the narrow sense) of a system of algebras of the same type $[A_\alpha]_{\alpha\in M}$ we mean the algebra of the same type whose elements are systems of elements (components) a_α, one from each A_α, with only a finite number of components of each element different from the distinguished elements of the corresponding algebras. Operations on elements are defined componentwise. The direct product is $\prod_{\alpha\in M} A_\alpha$.

In what follows we shall speak only of algebras of the class under consideration.

Lemma 1. *A subalgebra H is a normal divisor of an algebra A if and only if it satisfies the requirements:*

1. $\chi(x,h)\in H \leftrightarrow x\in H$, where $h\in H$.
2. $\chi(x,y)\in H \rightarrow \chi(y,x)\in H$.
3. $\chi(x,y)\in H \;\&\; \chi(y,z)\in H \rightarrow \chi(x,z)\in H$.
4. $\chi(a_1,b_1)\in H \;\&\; \dots \;\&\; \chi(a_n,b_n)\in H \rightarrow \chi(f(a_1,\dots,a_n), f(b_1,\dots,b_n))\in H$, where f is an operation of the algebra A.

Let H be a normal divisor, and φ a homomorphism with kernel H. Let us show how the fulfillment of conditions 1-4 is verified in example 2: if $\chi(x,y)\in H$, then $\chi(x,y)\varphi=\chi(x\varphi,y\varphi)=e$, i.e. $x\varphi=y\varphi$, and consequently $\chi(y\varphi,x\varphi)=e$, i.e. $\chi(y,x)\in H$.

Conversely, let H be a subalgebra satisfying requirements 1-4. We shall say that $x\sim y$ if and only if $\chi(x,y)\in H$. The relation $x\sim y$ is a congruence. The kernel of the natural homomorphism of the algebra A onto the factor algebra by this congruence, in accordance with requirement 1, is H.

Remark. There exists a normal divisor H, maximal among those satisfying the requirement: $x\notin H\supset K$, where K is a normal divisor and $x\notin K$.

Indeed, the union of an increasing chain of normal divisors with the indicated property is a normal divisor (a subalgebra satisfying the requirements of the lemma). We apply Zorn' s lemma.

Lemma 2. *Under a homomorphism, the image of a normal divisor is a normal divisor.*

Let H be the kernel of φ; N the kernel of ψ; P the full ψ-preimage of $H\psi$; Q the full φ-preimage of $N\varphi$. Then

$$P:\{p\mid p\equiv h\equiv e\},\qquad Q:\{q\mid q\equiv n\equiv e\}.$$

Here the congruence signs are taken with respect to ψ,φ and φ,ψ, respectively. Let $p\in P$, $h\in H$, and $h\psi=p\psi$. Then

$$p=\chi(p,e)\equiv\chi(p,h)\equiv\chi(p,p)=e,$$

with respect to φ and ψ, respectively, i.e. $P \subset Q$. Similarly $Q \subset P$. Consequently, $P = Q$. We shall say that $a\,\theta\,b$ if and only if there exist chains

$$a \equiv a_1 \equiv b_1 \equiv \cdots \equiv b$$

and

$$a \equiv a_1' \equiv b_1' \equiv \cdots \equiv b.$$

Here the equivalences are taken alternately with respect to φ and ψ. The relation θ is a congruence. The natural homomorphism of the algebra A onto A/θ has kernel P. We prove that $H\psi$ is a normal divisor. We show, using condition 2 as an example, the fulfillment of the conditions of Lemma 1 for $H\psi$. If $\chi(\bar{x}, \bar{y}) \in H\psi$, then $\chi(x, y) \in P$, where $x\psi = \bar{x}$, $y\psi = \bar{y}$; $\chi(y, x) \in P$, i.e. $\chi(\bar{y}, \bar{x}) \in H\psi$. Here we use the fact that P is a normal divisor.

Lemma 3. *A normal divisor N of an algebra A is maximal if and only if the image of A under a homomorphism whose kernel is N is a simple algebra.*

Let N be the kernel of the homomorphism φ, and let $A \supset L \supset N$, where L is a normal divisor, the kernel of a homomorphism ψ. Let $A\varphi = X$ be a simple algebra and $A\psi = Y$. To each element of X we put in correspondence that element of Y which is the image, under the mapping ψ, of the preimage of the original element under the homomorphism φ. This correspondence is single-valued: if $a\varphi = b\varphi$, then $e = \chi(a\varphi, b\varphi) = \chi(a, b)\varphi$, i.e. $\chi(a, b) \in N \subset L$, and

$$e = \chi(a, b)\psi = \chi(a\psi, b\psi), \qquad a\psi = b\psi.$$

This correspondence is a homomorphism. The homomorphic image Y of the simple algebra X may be either e or isomorphic to it. In the first case $L = A$, in the second $L = N$.

Indeed, let $l \in L \setminus N$. Then $l\varphi \neq e$, $l\psi = e$, i.e. the constructed mapping, being an isomorphism, sends a nonsingular element to a singular one, which is impossible. Conversely, if $A\varphi = X$ is not a simple algebra, then there exists a homomorphism σ of the algebra X such that the homomorphism $\varphi\sigma$ of the algebra A does not send A only into e, and its kernel is larger than the kernel of φ, i.e. N is not maximal.

Theorem 1. *Every nontrivial normal divisor of an algebra A is contained in a proper direct factor if and only if A is equal to a direct product (in the narrow sense) of simple subalgebras.*

Suppose every nontrivial normal divisor H of the algebra A is contained in a proper direct factor of the algebra A, and let $\mathfrak{S} = (S_\alpha)_{\alpha \in M}$ be the totality of all simple normal divisors of the algebra. Let $K = (S_\alpha)_{\alpha \in M}$ be the normal divisor generated by all S_α. Repeating the arguments carried out in (1), one can show that $K = A$.

Choose all subsets $\widetilde{M}$ of the set M which are one-element sets, or such that any two disjoint nonempty finite parts $\widetilde{M}_1$ and $\widetilde{M}_2$ of the set $\widetilde{M}$ have the property

$$(S_\alpha)_{\alpha \in \widetilde{M}_1} \cap (S_\alpha)_{\alpha \in \widetilde{M}_2} = e.$$

The collection of such sets $\widetilde{M}$ is partially ordered by inclusion. The union of the elements of an increasing chain has the same property and, consequently, by Zorn' s lemma, there exists a maximal subset M_1 of the set M with this property. We shall prove that

$$(S_\alpha)_{\alpha \in M_1} = A.$$

Suppose the contrary. Then $(S_\alpha)_{\alpha \in M_1} = B$ is a nontrivial normal divisor of the algebra A. By hypothesis, there exists a true direct factor B' containing B:

$$A = B' \times C.$$

In the normal divisor C there is at least one simple normal divisor $P \neq e$ of the algebra A. Indeed, not all $[S_\alpha]_{\alpha \in M}$ can be contained in B'; therefore for at least one α,

$$S_\alpha \cap B' = e, \qquad S_\alpha \neq e.$$

Then the homomorphism which sends each element of the direct product to the same element but with first component e sends S_α into P—a normal divisor of the algebra C (by Lemma 2) and, as is easily seen, a normal divisor of the algebra A. P is a simple normal divisor, since it is the image of the simple algebra S_α. Moreover,

$$B \cap P = e, \qquad P \neq e,$$

i.e. the number P in the enumeration M does not belong to M_1.

The normal divisor generated by $(S_\alpha)_{\alpha \in \widetilde{M}_2}$ and P contains exactly those elements whose first component is from $(S_\alpha)_{\alpha \in \widetilde{M}_2}$, and whose second is from P. The normal divisor generated by $(S_\alpha)_{\alpha \in \widetilde{M}_1}$ contains exactly those elements whose first component is from $(S_\alpha)_{\alpha \in \widetilde{M}_1}$, and whose second is e. These two normal divisors intersect in e. We have obtained a contradiction to the fact that M_1 is a maximal set with the indicated property. Consequently,

$$(S_\alpha)_{\alpha \in M_1} = A.$$

It now remains only to show that the normal divisor generated by any finite subset of the simple normal divisors S_α, $\alpha \in M_1$, is equal to the direct product of simple subalgebras.

Let

$$R = (S_{\alpha_1}, \dots, S_{\alpha_n}),$$

where $\alpha_i \in M_1$. Take the normal divisor

$$L_1 = (S_\alpha)_{\alpha \in M_1,\ \alpha \neq \alpha_1}.$$

It is true and, by Lemma 3, maximal. By hypothesis, it is contained in a true direct factor and, by maximality, coincides with it:

$$A = L_1 \times P_1,$$

where P_1 is a simple normal divisor.

Let

$$L_2 = (S_\alpha)_{\alpha\in M_1,\ \alpha\neq\alpha_1,\alpha_2}.$$

Then

$$L_1 = (L_2, S_{\alpha_2}),$$

the subalgebra $L_2 \times P_1$ is a maximal normal divisor of the algebra A; therefore, as above,

$$A = L_2 \times P_1 \times P_2,$$

where P_2 is a simple normal divisor. Continuing this decomposition further, we obtain

$$A = L_n \times P_1 \times \cdots \times P_n,$$

where P_i are simple normal divisors and

$$L_n = (S_\alpha)_{\alpha\in M_1,\ \alpha\neq\alpha_i} \quad (i = 1, \dots, n).$$

Let the canonical isomorphism existing between A and the direct product be denoted by ψ, and let the homomorphism with kernel L_n sending an element of the direct product to the same element but with first component e be denoted by φ. Then $\psi\varphi$ is an isomorphic mapping of R onto $P_1 \times \cdots \times P_n$, i.e. R is isomorphic to the direct product of simple algebras.

Conversely, let

$$A = \prod_{\alpha\in M_1} P_\alpha,$$

where P_α are simple, and let H be a nontrivial normal divisor of the algebra A.

Consider the collection of all simple normal divisors $[S_\alpha]_{\alpha\in M}$ of the algebra A, and single out those subsets Q of the set M which satisfy the condition

$$H \cap (S_\alpha)_{\alpha\in Q} = e.$$

By Zorn' s lemma we establish the existence of a maximal set Q_1 with this property. Let

$$T = (S_\alpha)_{\alpha\in Q_1},$$

and let H' be a normal divisor, maximal among those which have the property

$$T \cap H' = e, \qquad H' \supset H.$$

Let φ be a homomorphism of the algebra A with kernel T; let S be some simple normal divisor of the algebra A, $S \not\subset T$. Then

$$S\varphi \subset H\varphi \subset H'\varphi,$$

since $S\varphi$ is a simple normal divisor, and otherwise a homomorphism δ of the algebra $A\varphi$ with kernel $S\varphi$ would send $H\varphi$ into $H\varphi\delta$, and moreover any element $h \in H,\ h \neq e$, under the mapping $\varphi\delta$, would be sent to

$$h\varphi\delta \neq e,$$

i.e. H would not intersect the kernel of $\varphi\delta$, which contains-

by (T, S), which is impossible (by the maximality of Q_1). Consequently, $A\varphi = H'\varphi$.

Similarly it is proved that $A\psi = T\psi$, where ψ is a homomorphism with kernel H'. The homomorphisms φ and ψ induce isomorphisms φ_1 and ψ_1 of the algebra H' onto $H'\varphi$ and of T onto $T\psi$, respectively.

Form the direct product $H' \times T$. To each element $a \in A$ assign the element $(a\varphi\varphi_1^{-1}, a\psi\psi_1^{-1})$. This mapping of A into $H' \times T$ is a homomorphism. Its kernel is the intersection of the kernels of φ and ψ, i.e. e. We shall show that this mapping is an isomorphism of A onto $H' \times T$. Elements of the form (h, e) and (e, t) have preimages $h \in H'$ and $t \in T$, respectively. We shall show that an element of the form (h, t) also has a preimage. Since $H' \times T$ is an algebra of the class under consideration, by definition there exists a function θ such that

$$\theta((h, e), (e, t)) = (\theta(h, e), \theta(e, t)) = (h, t).$$

It is clear from this that a preimage of (h, t) is $\theta(h, t)$. Consequently, the algebra A is equal to the direct product of H' by T, with $H \subset H'$.

By a subdirect product of a family of algebras $[S_\alpha]_{\alpha \in M}$ we shall mean a subalgebra S of the direct product $\prod_{\alpha \in M} S_\alpha$ such that for every element $s_\alpha \in S_\alpha$ there exists at least one element $s \in S$ whose α-component is s_α.

Theorem 2. *An algebra A is isomorphic to a subdirect product of a family of simple algebras if and only if, for every normal divisor N of the algebra A, distinct from the zero element, there exists a true normal divisor N' of the algebra A for which $(N, N') = A$.*

The proof is analogous to the proof of Theorem 2 in (1).

Corollary. *If every normal divisor of the algebra A, distinct from e, contains a direct factor distinct from the zero one, then the algebra A is isomorphic to a subdirect product of simple algebras.*

This follows from the assertion of the theorem and from the fact that if $A = P \times H$, then $A = (P, H)$.

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REFERENCES

1. T. J. Head, Proc. Am. Math. Soc., **15**, No. 2 (1964).

Note: Figure translations are in progress. See original paper for figures.