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1965

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Abstract

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MULTIDIMENSIONAL KNOTS

(Presented by Academician A. N. Kolmogorov, January 11, 1965)

1. In the present note we consider topological embeddings of the $(n-2)$-dimensional sphere S^{n-2} in n-dimensional Euclidean space E^n. In the first part a definition is proposed of a knot $S^{n-2} \subset E^n$, $n \geqslant 3$, and an equivalence relation between knots is introduced. The classes of mutually equivalent knots form a set Σ_n, which for $n = 3$ coincides with the usual set of isotopy classes of classical knots. Their theory is constructed, for $n > 3$, in parallel with the three-dimensional one, but has its own special features. In Σ_n there is naturally defined an operation of addition (“connected sum”), which endows this set with the structure of an abelian semigroup. It turns out that in the semigroup all elements except zero (the zero is the geometric sphere $S_0^{n-2} \subset E^n$) have no inverses, and every element decomposes into a finite sum of “prime” summands (i.e., into a sum of such nonzero elements of Σ_n, each of which can no longer be represented as a sum of two nonzero elements). The proofs, which are only outlined here, were carried out by the author for $n \geqslant 4$; for $n = 3$ analogous results had earlier been obtained by Schubert ([1]) by means of three-dimensional methods.

In the second part we describe general methods for constructing multidimensional analogues of wild simple closed curves and arcs in E^3 of Artin and Fox ([2]), applicable also for $n = 3$, and in this case examples are obtained that differ from those of Artin and Fox.

2. We list the necessary definitions. A **pair** of topological spaces (X, A) is a collection of two topological spaces, if $X \supset A$. The **equivalence** of pairs (X, A) and (Y, B) means the existence of a homeomorphism of pairs
$h : (X, A) \to (Y, B)$, i.e., of such a homeomorphism h, defined on X, that $h(X) = Y$ and $h(A) = B$. The pair (E^n, M^k) is called **tame** if it is equivalent to the pair $(E^n, \mathfrak{M}^k)$, where $\mathfrak{M}^k$ is a rectilinear polyhedron with respect to the affine structure of E^n. Non-tame pairs are called **wild**. The pair (E^n, M^k) is called **locally tame** at a point $p \in M^k$ if there exists a neighborhood $U \supset p$ such that the pair $(U, U \cap M^k)$ is equivalent to the pair (P^n, P^k), where P^n and P^k are polyhedra in E^n. The pair (E^n, M^k) is called **locally flat** at a point $p \in M^k$ if there exists a neighborhood $U \supset p$ such that the pair $(U, U \cap M^k)$ is equivalent to the pair (E^n, E^k), where E^k is a hyperplane in E^n. For $n = 3$ it is known that the pair (E^n, S^k) is locally tame if and only if it is locally flat;

however, for $n \geqslant 4$ the matter is different. If one takes an ordinary polyhedral knot S^1 lying in a hyperplane $E^3 \subset E^4$, and constructs the double cone with vertices at points p, p' lying in different half-spaces with respect to E^3, then the resulting sphere $S^2 \subset E^4$ will be polyhedral (and hence tame and locally tame), but will not be locally flat at the points p and p', since the local fundamental group of $E^4 \setminus S^2$ at the points p and p' is isomorphic to the group $\pi_1(E^3 \setminus S^1)$. This well-known example is easily generalized to pairs (E^n, S^{n-2}) for any $n \geqslant 4$. On the other hand, from the fulfillment

from the condition of local flatness there obviously follows the fulfillment of the condition of a locally tame embedding.

A homeomorphism $h : E^n \to E^n$ is called **stable** if it can be decomposed into a composition of several homeomorphisms, each of which is fixed on some region E^3 ([3]). For a stable homeomorphism $h : E^n \to E^n$ there exists an isotopy $h_t : E^n \to E^n$ such that $h_0 = 1$ and $h_1 = h$ (Alexander' s theorem, see ([4]), p. 345).

3. **The semigroup of knots Σ_n.** We shall call a locally flat pair $\sigma = (E^n, S^{n-2})$ a **knot** if S^{n-2} is oriented and contains a simplex $\Delta \in T(S^{n-2})^*$ which can be carried by a stable homeomorphism of E^n onto itself into a rectilinear one (relative to E^n). Two knots σ and τ are **equivalent** (notation $\sigma = \tau$) if by a stable homeomorphism of E^n onto itself one can be carried to the other with preservation of the orientation of the knots. The set of equivalence classes is denoted by Σ_n. By a knot we shall sometimes mean a concrete embedding, and sometimes an equivalence class—which meaning is intended will be clear from the context.

For what follows we need the following propositions asserting the existence of certain stable homeomorphisms:

A. *If $S^{n-2} \subset E^n$ is locally flat, $g : S^{n-2} \to S^{n-2}$ is a stable homeomorphism, and U is any neighborhood of S^{n-2}, then g can be extended to a homeomorphism of all of E^n, fixed outside U.*

B. *If Δ is a simplex of the standard (curvilinear) triangulation of a knot $S^{n-2} \subset E^n$, and U is any neighborhood of Δ in E^n, then there exists a homeomorphism h, fixed outside U and such that $h(\Delta)$ is a rectilinear simplex relative to the affine structure E^n.*

Consider two knots $\sigma_1, \sigma_2 \in \Sigma_n$ and define their connected sum $\sigma_1 \# \sigma_2$. To this end choose representatives (E^n, S_1^{n-2}) and (E^n, S_2^{n-2}) of these knots, separated by a hyperplane E^{n-1}. On each sphere S_i^{n-2} choose a simplex $\widetilde{\Delta}_i^{n-2} \in T(S_i^{n-2})$ and, using Proposition B, straighten $\widetilde{\Delta}_i^{n-2}$ in a small neighborhood of it by a homeomorphism φ_i. Let Δ_i^{n-2} denote a straight simplex lying in $\varphi_i(\widetilde{\Delta}_i^{n-2})$, concentric with $\varphi_i(\widetilde{\Delta}_i^{n-2})$, and let L be a finite broken line joining the barycenters of the simplexes Δ_1^{n-2} and Δ_2^{n-2}, intersecting $S_1^{n-2} \cup S_2^{n-2}$ only at its endpoints and intersecting the plane E^{n-1} in only one point. It is easy to see that there exists a homeomorphism

$$f : \Delta^{n-2} \times [1,2] \to E^n$$

such that

$$f(\Delta^{n-2} \times \{i\}) = \Delta_i^{n-2}, \qquad f(\{O\} \times [1,2]) = L$$

(where O is the center of Δ^{n-2}),

$$f(\Delta^{n-2} \times [1,2]) \cap (S_1^{n-2} \cup S_2^{n-2}) = \Delta_1^{n-2} \cup \Delta_2^{n-2},$$

and the orientations of the spheres S_1^{n-2} and S_2^{n-2} are compatible relative to the "connecting strip"

$$T = f(\partial\Delta^{n-2} \times [1,2]).$$

We put (by definition)

$$\sigma_1 \# \sigma_2 = (E^n, (S_1 \cup S_2 \setminus (\Delta_1 \cup \Delta_2)) \cup T);$$

the orientation of the sum is determined by the orientation of any summand.

It turns out that this definition is correct, i.e. depends only on the classes $\sigma_1, \sigma_2 \in \Sigma_n$ and not on the choice of representatives and of the construction described above. This fact is proved with the help of Proposition A, certain theorems of Brown and Gluck, and geometric lemmas which we do not formulate here. The proof also goes through in the case $n = 3$.

Theorem 1. *The connected-sum operation endows the set Σ_n of knot classes (for $n \geq 4$) with the structure of an abelian semigroup. The zero of the semigroup Σ_n is the trivial knot (the class of the geometric sphere $(E^n, \partial\Delta^{n-1})$), and all its nonzero elements have no inverses.*

The proof of commutativity presents no great difficulties after correctness has been proved. The nonexistence of inverse elements is proved by means of Fox' s method (construction of an infinite sum (see ([8]), p. 142).

* Every sphere $S^k \subset E^n$ we regard as a homeomorphic image of the boundary of the simplex Δ_0^k; a curvilinear triangulation of S^k, which is the image of some subdivision of Δ_0^k under the homeomorphism under consideration, is denoted by $T(S^k)$.

4. Let $\sigma_1, \sigma_2, ... \in \Sigma_n$. One can define the **infinite connected sum** $\mathop{\#}\limits_{i=1}^{\infty} \sigma_i$ in the following way. Let B be an n-dimensional cone with vertex p and base B_0^{n-1}. Divide B into truncated cones $B_1, B_2, ...$ by means of $(n-1)$-dimensional hyperplanes $H_1, H_2, ...,$ parallel to B_0^{n-1} and converging to the hyperplane $H_\infty \ni p$. Consider inside each B_i a copy* (E^n, S_i) of the knot σ_i. Then, for each i, realize the connected sums $\sigma_i \# \sigma_{i+1}$ within the set $B_i \cup B_{i+1}$ in such a way that the connecting strips do not intersect one another and intersect the corresponding H_i only once. If to the resulting set we add the limit point p, we obtain a set which we denote by S_∞. It

is easy to see that this set is an $(n-2)$-dimensional sphere, and moreover locally flat at all points except, perhaps, the point p. We put, by definition,

$$\mathop{\#}_{i=1}^{\infty} \sigma_i = (E^n, S_\infty).$$

It turns out that this definition is also correct, i.e., it depends neither on the order of the spheres σ_i, nor on the choice of representatives, nor on the arbitrariness of the constructions described above.

However, it should be noted that infinite summation, generally speaking, takes one outside the class Σ_n, since S_∞ may fail to be locally flat at the point p, as the following basic result shows.

Theorem 2. *If $n \geq 4$, then the infinite connected sum $\mathop{\#}_{i=1}^{\infty} \sigma_i = (E^n, S_\infty)$ of nontrivial knots $\sigma_1, \sigma_2, \ldots \in \Sigma_n$ is not locally tame (and consequently not locally flat) at the limit point $p \in S_\infty$.*

The proof of the theorem is based on applying Schreier' s theorem to the computation of the fundamental group of the set $E^n \setminus S_\infty$: in a neighborhood of the point p this group has an infinite set of generators, which cannot occur if S_∞ is locally tame at the point p.

5. We call a knot $\sigma \in \Sigma_n$ **simple** if it cannot be represented in the form $\sigma = \sigma' \# \sigma''$, where $\sigma', \sigma'' \neq 0$.

Theorem 3. *Every knot $\sigma \in \Sigma_n$ can be decomposed into a finite sum of simple knots.*

The existence of a finite decomposition is proved by contradiction: assuming the impossibility of a finite decomposition for some knot $\sigma \in \Sigma_n$, we obtain a chain of equalities

$$\sigma = \sigma_1 \# \sigma_2', \quad \sigma_2' = \sigma_2 \# \sigma_3', \quad \sigma_3' = \sigma_3 \# \sigma_4', \ldots,$$

where $\sigma_i \neq 0$ and the σ_i' are not simple. Applying one more proposition on stable homeomorphisms, one can show that then there is a point (a limit point for the knots σ_i) at which σ is not locally flat.

6. **Examples of wild embeddings S^{n-2} in E^n.**

I. From Theorem 2 it follows that there exist wild embeddings $S^k \subset E^n$ ($k = n-2,\ n \geq 4$) for which the set of points of non-local flatness is any finite (or countable compact) set, as well as a Cantor discontinuum lying on an interval. From Chernavskii' s results it follows that such constructions are impossible for $k \neq n-2$, if $n \geq 5$.

II. For some knots $\sigma \in \Sigma_n$, the infinite connected sum

$$\mathop{\#}_{i=1}^{\infty} \sigma_i = (E^n, S_\infty),$$

where $\sigma_i = \sigma$ for all i, has the following property: if one removes from S_∞ an open simplex $\operatorname{Int}\Delta^{n-2} \subset S_1$ (where $(E^n, S_1) = \sigma_1$), then one obtains an $(n-2)$-dimensional disk D^{n-2}, simultaneously cellular** and wild (not locally tame at one point).

III. For some knots $\sigma \in \Sigma_n$ (namely for those knots which admit canonical cuts, defined below), the infinite connec-

* Some additional requirements are imposed on the choice of $(E^n, S_i) \in \sigma_i$.

** $Q \subset E^n$ is cellular (5) if in every neighborhood of the set Q one can inscribe a topological n-dimensional cube D^n containing Q.

the connected sum $\mathop{\#}\limits_{i=1}^{\infty} \sigma_i = (E^n, S_\infty)$, where $\sigma_i = \sigma$ for all i, makes it possible to construct a wild $(n-2)$-dimensional disk (with a single point of non-local flatness) $D = S_\infty \setminus \operatorname{Int}\Delta^{n-2}$, which decomposes into a pair of trivial* disks $D_1 \cup D_2 = D$, whose intersection is a trivial $(n-3)$-dimensional disk ($n \geq 4$). Similar pairs of k-dimensional disks, if $k \neq n-2$, do not exist, as A. V. Chernavskii has shown ([7]).

To describe the indicated example and to define the canonical cut, the following notation is needed. Let H_0 be an $(n-1)$-dimensional hyperplane in E^n; $p \notin H_0$ a point; $\Delta^{n-2} \subset H_0$ a disk bounded (in H_0) by the sphere S^{n-3}; S^{n-4} the equator of the sphere S^{n-3}, dividing S^{n-3} into two closed hemispheres Δ_1 and Δ_2. Let χA, where $A \subset E^n$, be the union of all segments ap, where $a \in A$. Divide the cone $B = \chi\Delta^{n-2}$ into truncated cones $B_1, B_2, \ldots$ by means of hyperplanes $H_1, H_2, \ldots$, parallel to H_0 and converging to the plane $H_\infty \ni p$. The cone thus constructed will serve as the basis for constructing the infinite connected sum $\mathop{\#}\limits_{i=1}^{\infty} \sigma_i$. Namely, we choose the representative (E^n, S_i) of the knot σ_i so that the sphere S_i coincides with ∂B_i everywhere outside a certain n-dimensional ball containing the whole "knotted part" S_i and lying between H_{i-1} and H_i. Then, in order to construct the connected sum, there is no need to construct connecting tubes; it suffices to remove the "interior" of the common part of the knots S_i and S_{i-1}, i.e., the set $G_i = \chi(\Delta^{n-2} \setminus \partial\Delta^{n-2}) \cap H_i$.

Consider any knot S_i. A representation of the set $S_i \setminus G_i \cup G_{i+1}$ as the union of two $(n-2)$-dimensional disks $D_{i1} \cup D_{i2}$ is called a **canonical cut** if: 1) $D_{ij} \cap H_{i-1} = H_{i-1} \cap \chi\Delta_j$, $D_{ij} \cap H_i = H_i \cap \chi\Delta_j$, $D_{i1} \cap D_{i2}$, homeomorphic to the cylinder $S^{n-3} \times [0,1]$, and moreover $(D_{i1} \cap D_{i2}) \cap H_i = \chi S^{n-3} \cap H_i$ and $(D_{i1} \cap D_{i2}) \cap H_{i-1} = \chi S^{n-3} \cap H_{i-1}$; 2) there exist homeomorphisms $h_1^{(i)}, h_2^{(i)}$ such that $h_j^{(i)}(D_{ij}) = \chi\Delta_i \cap B_i$, and the support of the homeomorphism $h_j^{(i)}$ lies strictly between H_{i-1} and H_i and inside the $1/i$-neighborhood of B_i ($j = 1, 2$). More briefly, but less precisely, this condition may be expressed as follows: a knot entering as a summand in the connected sum can be cut into two disks, each of which separately can be straightened by a homeomorphism not affecting the remaining knots. Canonical cuts exist for many knots, for example for Artin spun knots ([6]).

Take an infinite set of knots $\{\sigma_i\}$ admitting a canonical cut, and construct their infinite connected sum by the method just described. Then the disks $D_1 = \bigcup_{i=1}^{\infty} D_{i1} \cup p$, $D_2 = \bigcup_{i=1}^{\infty} D_{i2} \cup p$ are each trivial separately (this is easy to prove by combining the homeomorphisms $h_j^{(i)}$); their intersection is also a trivial $(n-3)$-dimensional disk; but their union $D_1 \cup D_2 = S_\infty \setminus H_0$ is wild (it is not locally tame at one point by virtue of Theorem 2).

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Received
13 XII 1964

References

[1] H. Schubert, *Sitzber. Heidelberg. Akad. Wiss.*, **3** (1949).
[2] E. Artin, R. Fox, *Ann. Math.*, **49**, 979 (1948).
[3] M. Brown, H. Gluck, *Bull. Am. Math. Soc.*, **69**, No. 1, 51 (1963).
[4] F. Klein, *Geometrie*, 1936.
[5] M. Brown, *Bull. Am. Math. Soc.*, **66**, No. 2, 74 (1960).
[6] E. Artin, *Abh. Math. Sem. Hamburg. Univ.*, **4**, 174 (1926).
[7] A. V. Chernavskii, *DAN*, **158**, No. 1, 62 (1964).
[8] M. K. Fort (Ed.), *The Topology of 3-manifolds*, Cliff., 1962.

* A disk $D^k \subset E^n$ is **trivial** if there exists a stable homeomorphism $h : (E^n, D^k) \to (E^n, \Delta^k)$, where Δ^k is a rectilinear simplex.

Note: Figure translations are in progress. See original paper for figures.