ON THE STRUCTURAL ORDERABILITY OF ALGEBRAS AND RINGS

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Abstract

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ON THE STRUCTURAL ORDERABILITY OF ALGEBRAS AND RINGS

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For algebras (rings) of a certain class, conditions are given that are necessary and sufficient for the possibility of transforming them into f-algebras (f-rings), and conditions for the uniqueness of such a transformation. Conditions are considered under which complete or partial multiplication in a linear (additive) structure determines an order. The question is also considered of the possibility of transforming a linear set (group) with disjoint elements into a linear (additive) structure.

In the note only algebras over the field of real numbers are considered. Without special reservations we shall assume that all algebras and rings occurring below are associative and that all (in particular, f-algebras and f-rings*) contain no nilpotent elements.

Let X be an algebra, not necessarily commutative. By definition we shall call an element x disjoint from an element y and write xdy, if $xy = 0$. For the disjointness relation the following conditions are satisfied: 1) $xdy \iff ydx$; 2) $xdx \iff x = 0$; 3) xdy and λ real $\Rightarrow \lambda xdy$; 4) xdz, $ydz \Rightarrow (x+y)dz$; 5) xdy, $(x+y)dz \Rightarrow xdz$. Let $x \in X$. By X_x we shall denote the least annihilator of the annihilator of x. In X the condition is fulfilled: 6) $X_{xy} = X_{yx} = X_x \cap X_y$ for any $x, y \in X$. For the commutative case this was noted in ([5]).

In what follows we shall call an annihilator a component. This is justified by the fact that, in the case of an f-algebra Y, the set of annihilators coincides with the set of components of the K-lineal (linear structure) Y**. An annihilator of the form X_x will accordingly be called a principal component. Let X' and X'' be two mutually complementary components in X, $x \in X$; we shall say that there exists $x' = \mathrm{Pr}_{X'}\, x$—the projection of x onto X' (or a fragment of x), if a representation $x = x' + x''$ is possible, where $x' \in X'$, $x'' \in X''$ (such a representation can only be unique). This definition coincides with the usual definition of a projection in a K-lineal ([2]). An algebra in which there exists a projection of every element onto any component will be called an algebra with projections, and if there is only a projection onto any principal component—an algebra with projections onto a principal component. These definitions also coincide with the usual definitions for a K-lineal.

The set of all components in X, ordered by inclusion, forms a complete Boolean algebra, and $X_1 \wedge X_2 = X_1 \cap X_2$. Let $\{X_\xi\}$ be a complete system of pairwise disjoint components in X (completeness means that only the zero component in X can be disjoint from all X_ξ). If for every ξ there exists $x_\xi = \mathrm{Pr}_{X_\xi} x$, then we shall call x the union of the elements x_ξ and write $x = Sx_\xi$.

* A structurally ordered algebra (ring) X is called an f-algebra (f-ring) in the sense of G. Birkhoff and R. Pierce (^{4}), if $x, y, z \in X^+$, $x \wedge y = 0 \Rightarrow zx \wedge y = xz \wedge y = 0$.

** In a linear (additive) structure Y, elements x, y are called disjoint if $|x| \wedge |y| = 0$. A component is the set of all elements disjoint from every element of some subset of Y.

What was said above can also be carried over to the case of a ring X. One need only replace 3) by the condition: $3'$) $x\,d\,y \Rightarrow -x\,d\,y$. In a ring without torsion, moreover, $3''$) $x\,d\,y \Rightarrow \frac{1}{n} x\,d\,y$, if $\frac{1}{n}x$ exists (n natural).

Under what conditions can a structural order be introduced in an algebra (ring) X, making X an f-algebra (f-ring)? When can this be done in a unique way? Previously only the question was posed of under what conditions X can be completely ordered (in the absence of zero divisors in X; see (6)) or represented in the form of an algebra or a ring (but not an f-algebra and f-ring) of functions (7). We shall give an answer to the questions posed for the case of a commutative algebra (commutative ring without torsion*) with projections. First we formulate the following result, which probably also has independent significance.

Theorem 1. *Let X be an algebra (an f-algebra). Then there exist algebras (f-algebras) with projections containing X as a subalgebra (f-subalgebra). Among all such algebras (f-algebras) there exists a smallest $\overline{X}$, in the sense that if Y is any one of such algebras (f-algebras), then there exists a certain subalgebra (f-subalgebra) of it containing X and isomorphic to $\overline{X}$ under an isomorphism leaving the elements of X fixed. Analogous assertions hold for a ring (f-ring) X.*

We give the scheme for constructing $\overline{X}$. Consider all possible formal finite "sums" of symbols $\sum_k [X_k] x_k$, where $(X_1, \ldots, X_n)$ is some complete system of pairwise disjoint components in X, $\{x_k\} \subset X$, and the order of the "summands" is immaterial. We shall identify two such "sums" $\sum [X_k] x_k$ and $\sum [X'_i] x'_i$ if $(x_k - x'_i)\,d\,(X_k \cap X'_i)$ for all k, i; in particular, we shall identify the "sums" $\sum_k [X_k] y_k$ and $\sum_{k,i} [X_k \cap X'_i] y_k$. The classes of equivalent "sums" will be regarded as elements in $\overline{X}$. Algebraic operations in $\overline{X}$ are introduced "coordinatewise" (after reducing the "sums" to one system of components). In the case of an f-algebra (f-ring) X, we shall regard $\bar{x} \in \overline{X}^+$ if a representation of $\bar{x}$ with $\{x_k\} \subset X^+$ is possible. To an element $x \in X$ we associate in $\overline{X}$ the class containing the "sum" $[X]x$. This realizes an embedding of X in $\overline{X}$. The mapping $\overline{X}' \to X' \equiv \overline{X}' \cap X$ establishes a one-to-one correspondence between the sets of components in $\overline{X}$ and in X. Let $\overline{X}'_1, \overline{X}'_2$ be two mutually complementary components in $\overline{X}$, and let $\bar{x} \in \overline{X}$. Then there exists $\Pr_{\overline{X}_{1'}} \bar{x}$—namely, the class containing $\sum_{k,i}[X_k \cap X'_i]x_{ki}$, where $\sum_k [X_k]x_k$ is some representation of $\bar{x}$, $x_{k1} = x_k$, $x_{k2} = 0$.

Remark. The corresponding assertions remain valid for the case of an arbitrary K-lineal (commutative l-group) X.

Let now X be a commutative algebra. In general, all algebras and rings considered below will be assumed commutative. Suppose $x = yz$. In view of 6) one will have $X_x \subset X_y$ and $X_x \subset X_z$; if in fact $X_x = X_z$, we shall write $z = x/y$. If the quotient x/y thus defined exists, then it is uniquely determined.

Let now X be an f-algebra. The set X^+ of positive elements in X satisfies the following conditions ($x, y, x_\xi \in X^+$): a) if λ is a nonnegative number, then $\lambda x \in X^+$; b) $x + y \in X^+$; c) $xy \in X^+$; d) $x/y \in X^+$, if the quotient exists; e) $\Pr_{X'} x \in X^+$, if the projection exists; f) $Sx_\xi \in X^+$.

* Recall that an f-ring is a ring without torsion.

In the algebra X we shall call an arbitrary $H \subset X$ p-**closed** if it is closed with respect to the operations a)—e) (closure with respect to f) is not required!). For an arbitrary $B \subset X$ ($0 \in B$), put

$$X_B = \{x = \sum x_i : x_i \in B\},$$

$$X^B = \{x : \text{ if } x' \text{ is a fragment of } x \text{ and } x' \in -B, \text{ then } x' = 0\}.$$

If B is **proper** (i.e. $x, -x \in B \Rightarrow x = 0$), then X_B is proper and $X_B \subset X^B$; and if in X there exist projections onto the main component, B is p-closed, then X_B is also p-closed.

Let $Q = \{\sum x_i^2 : x_i \in X\}$, and let P be the smallest p-closed set in X containing Q. If in X there exist projections onto the main component, P has the form

$$P = \{\sum x_i^2 / \sum y_k^2 : x_i, y_k \in X\}.$$

Everything said after the remark to Theorem 1 carries over also to the case of rings. One need only replace condition a) by the condition: a′) $\frac{1}{n}x \in X^+$, if it exists (n natural).

Lemma. If $H \supset P$ is a proper p-closed set in the algebra (ring without torsion) X with projections onto the main component, and if $v \in X^H \setminus X_H$, then the set

$$H(v) = \left\{ \frac{x + yv'}{2} : x, y, z \in H, \ v' \text{ is a fragment of } v \right\}$$

is the smallest proper p-closed set containing H and v.

Theorem 2. In order that the algebra (ring without torsion) X with projections can be made into an f-algebra (f-ring), it is necessary and sufficient that the set Q be proper. In this case X_P coincides with the intersection of all cones of positive elements of possible structural orders in X, and X^P with their union.

It follows immediately that

Theorem 3. Under the conditions of Theorem 2, X can be made into an f-algebra (f-ring) in exactly one way if and only if

$$X_P = X^P.$$

Remark. Theorems 2 and 3 do not carry over to the case when in X there exist projections only onto the main component. In particular, the equality $X_P = X^P$ does not ensure the possibility of introducing a structural order in X. A corresponding algebra is, for example, the algebra of functions consisting of all real functions of the form

$$\sum_{k=0}^{n} \alpha_k t^k + \sum_{i=1}^{m} \beta_i \delta_{\tau_i}(t),$$

where $t \in (-\infty, +\infty)$, and $\delta_\tau(t) = 1$ for $t = \tau$ and 0 for $t \neq \tau$.

Theorem 3 can be considered directly for an f-algebra (f-ring) X. In this case it can be strengthened (in the sufficiency part).

Let X be a K-linear (commutative l-group) with **partial multiplication** (p.m.) in the sense of ([1]) (conditions 1)–6)). Recall that if the p.m. is **complete** (i.e. the product is defined for all pairs of elements), then this is equivalent to the assertion that X is an f-algebra (f-ring). One says that the p.m. **defines an order** in X if no other structural order compatible with the p.m. can be introduced in X. If the p.m. defines an order in X, then all information about X can, in principle, be obtained by considering only the algebraic operations in X.

In X, as a linear (additive) structure, the notion of disjointness is already present. Therefore the concepts introduced above for algebras (rings), in particular the concept of p-closedness, can also be transferred to the case under consideration.

Theorem 4. *Let X be a K-lineal (commutative l-group) with a.c. In order that the a.c. define an order in X, it is sufficient—and, in the case when multiplication is complete and there exist projections in X, also necessary—that the equality $X_P = X^P$ hold.*

From this the following results are easily obtained from ([1]).

Corollary. *In an f-algebra that is a K_σ-space, multiplication defines an order. A realizational a.c. in a K-space defines an order.*

We note that in both of these cases even the relation

$$X_Q = X^+ = X^Q$$

holds; moreover, any $x \in X^+$ is representable in the form of a countable sum $\sum x_n^2$. It is also interesting that in these cases X^+ can be obtained from all possible squares by means of the single operation of division:

$$X^+ = \{x^2/y^2 : \ x, y \in X\},$$

whence, in particular, $X^+ = P$.

Let us consider one more question. Let X be a **linear set (group) with disjoint elements** in the sense of V. I. Sorokin ([3]). This means that in X there is axiomatically defined a disjointness relation satisfying conditions 1)-5) (for the group case, incidentally, in this case, by definition, commutativity is assumed; instead of 3) one must take 3′)), and moreover in X there exists a projection onto any component. If X is a torsion-free group, we shall additionally assume that 3″) is fulfilled.

Theorem 5. *In a linear set (torsion-free group) X with disjoint elements one can introduce a structural order, i.e. make X a K-lineal (l-group), in such a way that disjointness in the structural sense coincides with the originally given disjointness. At the same time one may require that the members of any fixed system of pairwise disjoint elements in X turn out to be positive.*

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* An a.c. in an Archimedean K-lineal X is called **realizational** if it coincides with the natural multiplication of functions under some realization of X as a K-lineal of functions on an extremally bicompact space.