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ON REPRESENTING SMALL CATEGORIES

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Abstract

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MATHEMATICS

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ON REPRESENTING SMALL CATEGORIES

(Presented by Academician P. S. Novikov on 1 VII 1964)

1. Notation and definitions. The letters X, Y, Z (possibly with indices) always denote sets; the cardinality of a set X is denoted by $|X|$. A cardinality will be called **attainable** if it is smaller than the first unattainable (in the strong sense) cardinality.

If $R \subset X \times X$, then the pair (R, X) (and sometimes also R itself) is called a **relation** on X; the relation (R, X) is called **reflexive**, **symmetric**, etc., if R has the corresponding property in the usual sense. Instead of $(x, y) \in R$ one sometimes writes xRy. A mapping $f : X \to Y$, carrying X into a subset of Y, is understood as a triple (φ, X, Y), where $\varphi \subset Y \times X$ satisfies the well-known conditions; for brevity, sometimes no distinction is made in notation between f and

$$\varphi = \{(f(x), x) \mid x \in X\}.$$

If there are relations (R, X), (S, Y) and a mapping $f : X \to Y$, then f is called RS**-admissible** if xRx' implies $f(x)Sf(x')$, i.e. if $f \circ R \subset S \circ f$ in the sense of composition of relations, and **strongly admissible** if $f \circ R = S \circ f$.

If (R, X) is a relation, then $C(R, X)$ and, respectively, $C^*(R, X)$ denotes the semigroup of all RR-admissible (respectively strongly admissible) mappings; obviously, this is a semigroup with identity. The relation (R, X) is called **invertible** if $C(R, X)$ is a group with identity.

By $\mathfrak{R}$ (respectively $\mathfrak{R}^*$) is denoted the category whose objects are relations (R, X), and whose morphisms from (R, X) to (S, Y) are the RS-admissible (respectively strongly RS-admissible) mappings; $\mathfrak{R}$ may also be regarded as the category of directed graphs, with the morphisms being mappings that carry edges into edges. By $\mathfrak{R}_a$ is denoted the full subcategory of $\mathfrak{R}$ whose objects are the antireflexive (R, X); by $\mathfrak{R}_t$, the full subcategory of $\mathfrak{R}^*$ whose objects are transitive antireflexive antisymmetric relations. If A is a set, then by $A\mathfrak{R}$ is denoted the following category: the objects are families of relations of the form $\{(R_a, X);\ a \in A\}$; the morphisms from $\{(R_a, X);\ a \in A\}$ to $\{(S_a, Y);\ a \in A\}$ are those $f : X \to Y$ which are $R_{aS}a$-admissible for every $a \in A$. Obviously, if $|A| = 1$, then $A\mathfrak{R}$ is isomorphic to $\mathfrak{R}$. By $\mathfrak{S}_0(l)$ will be denoted the category of

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T_0-topological spaces, whose morphisms are taken to be local homeomorphisms (onto a space).

Let $\mathfrak{K}$ be a category. If the class $\mathfrak{K}'$ of its morphisms is a set, then $\mathfrak{K}$ will be called a **small category**; the cardinality of $\mathfrak{K}'$ will be denoted by $|\mathfrak{K}|$.

Let $\mathfrak{K}$ and Ω be categories. We shall say that the category Ω is **representable** in $\mathfrak{K}$ if it is isomorphic to a full subcategory of $\mathfrak{K}$.

Finally, let us introduce abbreviated notation for the following assertions (in which $\mathfrak{a}$ is a cardinality): $\mathcal{F}(\mathfrak{a})$: there exists an invertible (R, X) such that $|X| \geq \mathfrak{a}$; $\mathcal{M}(\mathfrak{a})$: for every family of relations

$\{(R_{aX}) : a \in A\}$, $|A| \leq \mathfrak{a}$, there exists a relation (R, Y) such that the semigroups $C(R, Y)$ and $\bigcap\{C(R_a, X) \mid a \in A\}$ are isomorphic; $\mathfrak{K}(\mathfrak{a})$: if $|A| \leq \mathfrak{a}$, then $A\mathfrak{R}$ is representable in $\mathfrak{R}$.

2. Basic constructions.

For a given relation (R, X) or family of relations $\{(R_a, X)\}$, relations $R^{(1)}, R^{(2)}, R^{(3)}$ are constructed (in an essentially unique way).

1) Let (R, X) be a relation on the set X. Put $X^{(1)} = X \cup R \cup U$, $U = \{u_1^1, u_2^1, u_1^2, u_2^2, u_3^2\}$. On the set $X^{(1)}$ define the relation $R^{(1)}$ as follows: $u_1^1 R^{(1)} u_2^1$; $u_i^2 R^{(1)} u_{i+1}^2$, $i = 1, 2$; $u_2^1 R^{(1)} u_1^1$; $u_3^2 R^{(1)} u_1^2$; for every $x \in X$ and $i = 1, 2$, $u_1^{iR^{(1)}} x$; for every $(x, y) \in R$, $x R^{(1)} (x, y)$, $(x, y) R^{(1)} y$.

2) Let (S_i, Y_i) $(i = 1, 2)$ be relations, A a set, $\varphi_i : A \to Y_i$ $(i = 1, 2)$ mappings such that there exist $y_i^0 \in Y_i \backslash \varphi_i(A)$; let $j_1, \dots, j_4$ be natural numbers, $U_k = \{u_k(1), u_k(2), \dots, u_k(j_k)\}$, $U = U_1 \cup U_2 \cup U_3 \cup U_4$. Suppose $\{(R_{aX}), a \in A\}$ is a family of relations on X. Then we put $Z = \{(x, x', a) \mid x R'_{ax},\ x, x' \in X\}$, $X^{(2)} = X \cup Y_1 \cup Y_2 \cup Z \cup U$, and define on the set $X^{(2)}$ the relation $R^{(2)}$ as follows:

$$u_k(i) R^{(2)} u_k(i+1), \qquad k = 1, 2, 3, 4, \quad i = 1, 2, \dots, j_k - 1;$$
$$u_k(j_k) R^{(2)} u_k(1), \qquad k = 1, 2, 3, 4;$$
$$u_k(1) R^{(2)} y_i, \qquad y_i \in Y_i, \quad i = 1, 2, \quad k = i, i+2;$$

for every $x \in X$, $i = 1, 2$, $y_i^0 R^{(2)} x$; for every $(x, x', a) \in Z$, $i = 1, 2$, $\varphi_i(a) R^{(2)} (x, x', a)$; for every $(x, x', a) \in Z$, $x R^{(2)} (x, x', a)$ and $(x, x', a) R^{(2)} x'$; for $y_i, y_i' \in Y_i$, $y_{iR}^{(2)} y_i'$ holds if and only if $y_{iS_{iy}} i'$.

3) Let (R, X) be a relation. Let $V = \{(x, y, i) \mid i = 1, 2;\ xRy\}$. Put $X^{(3)} = X \cup V$ and define $R^{(3)}$ on the set $X^{(3)}$ as follows: $(x, y, 1) R^{(3)} x$, $(x, y, 1) R^{(3)} y$, $(x, y, 1) R^3 (x, y, 2)$, $(x, y, 2) R^{(3)} y$.

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Proposition 1. If (R, X) is antireflexive, then $C(R, X)$ is isomorphic to $C(R^{(1)}, X^{(1)})$; moreover, $(R^{(1)}, X^{(1)})$ has no cycles of odd length not divisible by 3.

A **cycle of length** m for a relation (S, Y) is a sequence $x_1, x_2, \dots, x_n$ such that $x_{iSx_{i+1}}$, $x_{nSx}1$.

Proposition 2. Let (S_1, Y_1), (S_2, Y_2) be discretized relations, and let there be distinct prime numbers j_1, j_2, j_3, j_4 such that (S_i, Y_i) has no cycle of length j_k $(i = 1, 2;\ k = 1, \dots, 4)$. Let φ_i $(i = 1, 2)$ be a one-to-one mapping of the set A into Y_i, with $\varphi_i[A] \neq Y_i$. Suppose a family of relations $\{(R_a, X),\ a \in A\}$ is given such that for some a we have $R_a = \{(x, x) \mid x \in X\}$. Then, for $X^{(2)}, R^{(2)}$ defined above, the subsemigroup $C(R^{(2)}, X^{(2)})$ is isomorphic to $\bigcap\{C(R_a, X) \mid a \in A\}$.

Proposition 3. If (R, X) is antireflexive, then the semigroup $C[R, X]$ is isomorphic to $C_*[R^{(3)}, X^{(3)}]$.

3.

The main result of the paper is the following assertion:

Theorem 1. If $\mathfrak{a}$ is an attainable cardinal, $\mathfrak{K}$ a small category, $|\mathfrak{K}| \leq \mathfrak{a}$, then $\mathfrak{K}$ is representable in each of the categories $\mathfrak{R}$, $\mathfrak{R}_*$, $\mathfrak{R}_a$, $\mathfrak{R}_{f*}$, $\mathfrak{T}_0(l)$.

More generally, the following holds:

Theorem 2. Suppose that for the cardinal $\mathfrak{a}$ the condition $\mathcal{F}(\mathfrak{a})$ holds. Then every small category $\mathfrak{K}$ such that $|\mathfrak{K}| \leq \mathfrak{a}$ is representable in $\mathfrak{R}_a$, $\mathfrak{R}_t^*$, $\mathfrak{T}_0(l)$.

These theorems follow from the following assertions:

Theorem 3. Every small category $\mathfrak{K}$ is representable in the category $A\mathfrak{R}$, and as A one may take the set of morphisms of the category $\mathfrak{K}$.

Proof. Denote by K the set of objects of the category $\mathfrak{K}$, and by $M(a, b)$ the set of morphisms from the object a to the object b. Put $T(a) = (\bigcup\{M(b, a) \mid b \in K\};\ \{R_\alpha\},\ \alpha \in \mathfrak{K}')$, where $\beta R_\alpha \gamma \Longleftrightarrow \beta = \gamma \circ \alpha$; $T(\beta) = \{\gamma \mapsto \beta \circ \gamma\}$. It is easy to verify that this defines a mutu-

has a unique functor in $A\mathfrak{R}$. It remains to show that for every morphism $f : T(a) \to T(b)$ there exists $\varphi \in M(a, b)$, $f = T(\varphi)$. Let ε be the identity morphism of the object b. Since $a = \varepsilon \circ a$, it follows that $aR_\alpha \varepsilon$ and, consequently, $f(a) R_{\alpha f}(\varepsilon)$, or, by the definition of R_α, $f(a) = f(\varepsilon) \circ a$. Hence $f = T(f(\varepsilon))$.

Theorem 4. *Let $\mathcal{F}(|A|)$ hold. Then $A\mathfrak{R}$ is representable in $\mathfrak{R}_a$.*

Proof. Let (S_i, Y_i), $i = 1, 2$, be marked relations, $|Y_i| \geq |A|$. We may assume $|Y_i| > 1$; then the S_i are antireflexive. According to Proposition 1, there are distinct primes $j_1, \dots, j_4$ such that Y_i has no cycles of length j_k. We now carry out construction 2) for each object of the category $A\mathfrak{R}$, each time with the same (S_1, Y_1), (S_2, Y_2). It can be shown that the morphisms of the category $A\mathfrak{R}$ are put in one-to-one correspondence with the morphisms of the objects of the

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category $\mathfrak{R}$ obtained in the indicated way; more precisely, every morphism from $\{(R_a, X);\ a \in A\}$ to $\{(\overline{R}_a, \overline{X});\ a \in A\}$ is extended uniquely to a morphism from $(R^{(2)}, X^{(2)})$ to $(\overline{R}^{(2)}, \overline{X}^{(2)})$, and conversely, every morphism from $(R^{(2)}, X^{(2)})$ to $(\overline{R}^{(2)}, \overline{X}^{(2)})$ induces, for each $a \in A$, an $R_a\overline{R}_a$-admissible mapping.

Remark. As a consequence, we obtain the following equivalence of statements:

$$\mathcal{F}(a) \Longleftrightarrow \mathcal{M}(a) \Longleftrightarrow \mathcal{K}(a).$$

Theorem 5. *The category $\mathfrak{R}_a$ is representable in $\mathfrak{R}_t^*$.*

The proof is not difficult to give, using construction 3).

From Theorems 3 and 4, Theorem 2 follows directly. The assertion about the category $\mathfrak{L}_0(l)$ can be obtained by means of a modification of the result on $\mathfrak{R}_t^*$. Theorem 1 is a consequence of Theorem 2 and of the fact that, for attainable cardinal numbers, the assertion $\mathcal{F}(a)$ holds.

Let us note that, according to a personal communication from P. Vopěnka, $\mathcal{F}(a)$ holds even for some further cardinalities (of course, under the assumption of their existence).

Additional remarks. In particular, Theorems 1 and 2 can be applied to semigroups with identity. We obtain the assertion: every semigroup with identity and of attainable cardinality is isomorphic to some $C(R, X)$ and to some $C^*(S, Y)$.

By definition,

$$C(R, X) = \{f \mid f \circ R \subset R \circ f\}, \qquad C^*(R, X) = \{f \mid f \circ R = R \circ f\}.$$

In this connection, at first sight the question of representing semigroups in one of the following ways seems interesting (capital letters denote relations, and lowercase letters mappings):

A. $\{f \mid f \circ g \subset g \circ f\}$, in other words, $\{f \mid f \circ g = g \circ f\}$.

B. $\{S \mid S \circ f \subset f \circ S\}$.

C. $\{S \mid S \circ f = f \circ S\}$.

D. $\{S \mid S \circ R \subset R \circ S\}$.

E. $\{S \mid S \circ R = R \circ S\}$.

The answer, however, is negative, since even the permutation group of a set consisting of three elements cannot be represented in any of the ways A–E.

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Note: Figure translations are in progress. See original paper for figures.

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