Reports of the Academy of Sciences of the USSR

1965

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1965. Volume 164, No. 1

MATHEMATICS

Yu. A. Dmitriev

FREQUENCY CONDITIONS FOR DISSIPATIVITY AND THE EXISTENCE OF PERIODIC SOLUTIONS OF IMPULSE SYSTEMS WITH ONE NONLINEAR BLOCK

(Presented by Academician V. I. Smirnov, 13 II 1965)

We study the difference equations

$$x_{t+1} = Px_t + q\varphi(t, \sigma_t) + f(t, x_t), \qquad \sigma_t = r^* x_t, \quad t = 0, \pm 1, \pm 2, \dots, \quad (1)$$

which describe forced processes in impulse systems. In relations (1), x, q, r are real $\nu \times 1$-vectors; $\varphi(t, \sigma)$ and $f(t, x)$ are real functions, uniformly continuous in t, respectively scalar and $\nu \times 1$-vector functions of σ and x, defined for all σ and x; P is a nonsingular real $\nu \times \nu$ matrix whose eigenvalues π_j satisfy the conditions $|\pi_j| \neq 1,\ j = 1, \dots, \nu$ (with the exception of the case considered in Theorem 5); the asterisk denotes transposition.

A frequency condition for the absolute stability of forced processes in impulse systems with one nonlinearity was first derived by Ya. Z. Tsypkin (1). Theorem 2 (see below) supplements this result.

Definition. The system

$$x_{t+1} = g(t, x_t), \qquad t = 0, \pm 1, \pm 2, \dots, \quad (2)$$

where x and $g(t, x)$ are a real $\nu \times 1$-vector and a $\nu \times 1$-vector function, is called **dissipative** if in the space $\{x\}$ there exists a bounded closed set F possessing the properties: a) F is invariant for the system (2), i.e., from $x_{t_0} \in F$ it follows that $x_t \in F$ for $t > t_0$; b) F is a domain of attraction, i.e., for any $t_0,\ x_{t_0} \notin F$, there is a $t > t_0$ such that $x_t \in F$.

As in (2), denote:

$$\chi(\lambda) = r^*(P - \lambda I)^{-1} q, \qquad \chi_0 = \chi(0), \qquad \beta_j = \alpha_j(\alpha_j \chi_0 + 2),\ j = 1, 2,$$

$$\Phi(\lambda, \vartheta, \xi) = \mu_0^{-1} + \operatorname{Re} \chi(\lambda) - \vartheta[2 \operatorname{Re}(1 - \lambda)\chi(\lambda) + |(1 - \lambda)\chi(\lambda)|^2] +$$

$$+\xi|1 - \lambda|^2 [\chi_0 - 2 \operatorname{Re} \chi(\lambda) - \beta_0 |\chi(\lambda)|^2], \tag{3}$$

where the quantity α_0 may be assigned the values $\alpha_0 = \alpha_1$ or $\alpha_0 = \alpha_2$; the real numbers $\mu_0, \alpha_1, \alpha_2, \vartheta, \xi$ occur in the formulations of the theorems below, and the number β_0 is determined by the conditions: $\beta_0 = \beta_1$ if $(\beta_2 - \beta_1)(\chi_0 + \alpha_1^{-1}) \leq 0$; $\beta_0 = \beta_2$ if $(\beta_2 - \beta_1)(\chi_0 + \alpha_2^{-1}) \leq 0$.*

---

* It is easy to verify that in the case when $\alpha_2^{-1} \geq -\chi_0 \geq \alpha_1^{-1}$, both values $\beta_0 = \beta_1, \beta_2$ are possible. In this case formulas (3) determine 4 different functions $\Phi(\lambda, \vartheta, \xi)$, corresponding to the 4 possible values of the pair (α_0, β_0). If, however, $\chi_0 < -\alpha_2^{-1}$ or $\chi_0 > -\alpha_1^{-1}$, then β_0 assumes one value. In this case formulas (3) determine two different functions $\Phi(\lambda, \vartheta, \xi)$, corresponding to the values $\alpha_0 = \alpha_1$ and $\alpha_0 = \alpha_2$. In Theorems 3, 4, 5 it is assumed that the inequality $\Phi > 0$ ($\Phi \geq 0$) is satisfied for at least one of the indicated functions.

Theorem 1. Suppose that the following conditions are satisfied: a) $\lim_{|\sigma|\to\infty} \sigma^{-1}\varphi(t, \sigma) \times [1 - \mu_0^{-1}\sigma^{-1}\varphi(t, \sigma)] \geq 0$ uniformly in t, $\mu_0 > 0$; b) $\lim_{|x|\to\infty} |x|^{-1} f(t, x) = 0$ uniformly in t; c) $\Phi(\lambda, 0, 0) > 0$ for all λ, $|\lambda| = 1$.

Then:

I. System (1) is dissipative and has a solution $x_t = x_0(t)$, bounded for $-\infty < t < +\infty$.

II. If $\varphi(t, \sigma)$, $f(t, x)$ are T-periodic functions of t (T is an integer), then system (1) has a T-periodic solution $x_t = x_0(t)$.

Theorem 2. Suppose that the following conditions are satisfied: a) $f(t, x) \equiv f(t)$ and $\varphi(t, 0)$ are bounded functions of t, $-\infty < t < \infty$; b) for arbitrary σ_1, σ_2 the inequalities

$$0 \leq (\sigma_2 - \sigma_1)^{-1}[\varphi(t, \sigma_2) - \varphi(t, \sigma_1)] \leq \mu_0,$$

where $\mu_0 > 0$, hold; c) $\Phi(\lambda, 0, 0) > 0$ for all λ, $|\lambda| = 1$*.

Then the solution $x_0(t)$ indicated in Theorem 1 is unique** and, for all $t \geq t_0$ and any solution $x_t = x(t)$ of system (1), the estimate

$$|x(t) - x_0(t)| \leq \varkappa_0 \rho_0^{(t-t_0)} |x(t_0) - x_0(t_0)|,$$

holds, where $\varkappa_0, \rho_0$ are certain numbers, $\varkappa_0 > 0$, $1 > \rho_0 > 0$, depending on the parameters of system (1) and the number μ_0, and not depending on $t_0, x(t_0), \varphi(t, \sigma)$.

Theorem 3. Suppose that: a) the requirements a), b) of Theorem 1 are satisfied; b) the function $\varphi(t, \sigma) \equiv \varphi(\sigma)$ does not depend on t; c) for arbitrary σ_1, σ_2 the inequalities

$$\alpha_1 \le (\sigma_2 - \sigma_1)^{-1}[\varphi(\sigma_2) - \varphi(\sigma_1)] \le \alpha_2,$$

where $\alpha_1 \le 0,\ \alpha_2 \ge \mu_0 > 0$, hold; d)

$$\lim_{|\sigma|\to\infty} \sigma^{-2} \left| \int_0^\sigma \varphi \, d\sigma - \frac{1}{2}\sigma\varphi \right| = 0;$$

e) there exist numbers ϑ, ξ satisfying the conditions: $\vartheta \le 0$ when $\alpha_0 = \alpha_1$, $\vartheta \ge 0$ when $\alpha_0 = \alpha_2$,

$$0 \le \xi(\varkappa_0 + \alpha_1^{-1})^{-1}$$

when $\beta_0 = \beta_1$,

$$0 \le \xi(\varkappa_0 + \alpha_2^{-1})^{-1}$$

when $\beta_0 = \beta_2$, and

$$\Phi(\lambda, \vartheta, \xi) > 0$$

for all λ, $|\lambda| = 1$.

Then system (1) is dissipative and has a solution bounded for $-\infty < t < +\infty$.

Theorem 4. Suppose that the conditions of Theorem 3 are satisfied for the case when $\alpha_1 = 0$, $\alpha_0 = \alpha_2$, $\xi = 0$, $\vartheta \ge 0$, the function $\varphi(\sigma)$ is everywhere differentiable, and $f(t, x)$ is a T-periodic function of t (T is an integer).

Then system (1) has a T-periodic solution.

Theorem 5. Suppose that: a) in system (1) $f(t, x) \equiv 0$; b) the function $\varphi(t, \sigma) \equiv \varphi(\sigma)$ does not depend on t, is everywhere differentiable, and satisfies the conditions: $\varphi(0) = 0$, $0 < \varepsilon \le \sigma^{-1}\varphi(\sigma) \le \mu_0$, $\alpha_1 \le \varphi(\sigma)' \le \alpha_2$, where $\alpha_1 < 0$, $\alpha_2 \ge \mu_0 > \varepsilon$; c) for the eigenvalues of the matrix P the inequalities $|\pi_j| \le 1$, $j = 1, \dots, \nu$, hold; d) the spectrum of the matrix $P_\mu = P + \mu q r^*$ for some value of μ, $0 < \mu \le \mu_0$, is entirely located inside the unit circle.

If there exist numbers ϑ, ξ satisfying the conditions of Theorem 3 with the inequality $\Phi > 0$ replaced by $\Phi \ge 0$, then the solution $x_t \equiv 0$ of system (1) is asymptotically stable in the large.

For a given amplitude-phase characteristic $\chi(\lambda)$, the question of the existence of numbers ϑ, ξ satisfying the conditions of Theorems 3 and 5 can be solved graphically, analogously to (3).

Theorem 5 extends the results of (2) to the critical case. With its help one can show that the frequency condition $\Phi(\lambda, \vartheta, \xi) > 0$, derived in (2), is broader (if boundary cases in the parameter space are not considered) than the Cetté criterion (4). For the proof it suffices to consider system (1), in which the matrix P has eigenvalues $\pi_{1,2} = e^{\pm i\omega_1}$, $\pi_{3,4} = e^{\pm i\omega_2}$, $\omega_{1,2} \neq 0, \pi$, $|\pi_j| < 1$ for

* Condition c) of Theorem 2 is the frequency condition of Ya. Z. Tsypkin (1).

** It is not difficult to show that under conditions a), b) of Theorem 2, requirements a), b) of Theorem 1 are fulfilled automatically; consequently, the assertions of Theorem 1 in this case are certainly valid.

$j = 5, \dots, \nu$, and the number μ_0 in requirement b) of Theorem 5 is sufficiently small. In this case the frequency condition $\Phi(\lambda, \vartheta, \xi) \geqslant 0$ makes it possible to detect stability for a set of positive measure in the space of the parameters of system (1). The condition $\Phi(\lambda, \vartheta, 0) \geqslant 0$, however, is satisfied only on a set of measure zero. The assertion concerning the Popov criterion follows from the latter by continuity.

Lemma 1. Let, in system (2), $g(t, x)$ be a function of x uniformly continuous in t. Suppose that there exist continuous scalar functions $V(x)$, $\alpha(x)$, and a number $\xi_0 > 0$ satisfying the conditions:
a) $\alpha(x) > 0$ in the domain $|x| \leqslant \xi_0$; b) $\Delta V = V[g(t, x)] - V(x) \leqslant -\alpha(x)$ for $|x| \geqslant \xi_0$; c) $\lim_{|x|\to\infty} V(x) = +\infty$.

Then:

I. System (2) is dissipative and has a solution bounded for $-\infty < t < +\infty$.

II. The invariant set F appearing in the definition of dissipativity can be found in the form $F = E\{V(z) \leqslant \eta\}^*$, where η is some sufficiently large positive number.

Lemma 2. Suppose that system (2) is dissipative; the set F in the definition of dissipativity is convex; $g(t, x)$ is a function of x uniformly continuous in t, and a T-periodic function of t (T is an integer).

Then system (2) has a T-periodic solution.

Let us note that the set $F = E\{V(x) \leqslant \eta\}$ is convex if the matrix $\partial^2 V/\partial x^2$ of second derivatives is nonnegative.**

The proof of Lemma 1 can be carried out according to scheme (5). Lemma 2 is proved by applying Schauder' s principle (see, for example, (6)).

Proof of Theorem 1. Denote: $\varphi(t, \sigma_t) = \varphi_t$, $f(t, x_t) = f_t$. For the function $V(x) = x^* H x$ (where $H = H^*$ is a $\nu \times \nu$ matrix to be specified below), in view of equations (1), we have:

$$\Delta V = V(x_{t+1}) V(x_t) \equiv -\Psi(x_t, \varphi_t) - \Omega_1(\sigma_t^{-1}\varphi_t^2 + \Omega_2(f_t, x_t, \varphi_t).$$

In this expression

$$\Psi(x, \varphi) = x^* G x + 2g^* x\varphi + \gamma\varphi^2, \quad G = H - P^* H P, \quad -g = \tfrac{1}{2} r + P^* H q,$$

$$\gamma = \mu_0^{-1} - q^* H q, \quad \Omega_1(\mu) = \mu(1 - \mu_0^{-1}\mu), \tag{4}$$

$$\Omega_2(f, x, \varphi) = 2x^* P^* H f + 2q^* H f \varphi + f^* H f.$$

If condition c) of the theorem is fulfilled, then, by Lemma 2 of (2), one can choose in relations (4) a matrix $H > 0$ for which $\Psi(x, \varphi) > 0$ for all x, φ, $|x| + |\varphi| \neq 0$, and, consequently, there exist numbers $\varepsilon > 0$, $\varepsilon_1 > 0$ such that $\Psi(x, \varphi) \geqslant 2\varepsilon|x|^2 + \varepsilon_1|\varphi|^2 \geqslant 2\varepsilon|x|^2$. In view of conditions a), b) of Theorem 1, for the quantity Ω_1 one can obtain the estimate $\Omega_1\sigma^2 \geqslant -\frac{1}{2}\varepsilon|x|^2\omega_0$, where ω_0 is some positive number, and choose so large a $\xi_0 > 0$ that

$$|\Omega_2| \leqslant \tfrac{1}{2}\varepsilon|x|^2 + \omega_1|x|, \quad \omega_1 > 0, \quad \Delta V \leqslant -\varepsilon|x|^2 + \omega_1|x| + \omega_0 < 0$$

for $|x| \geqslant \xi_0$, and the function $V(x)$ satisfies all the conditions of Lemma 1, from which assertion I of Theorem 1 follows. Assertion II of the theorem follows at once from Lemma 2, since $\partial^2 V/\partial x^2 = H > 0$.

Proof of Theorem 2. If the conditions of Theorem 2 are fulfilled, the assertions of Theorem 1 hold. Let x_t', x_t'' be some arbitrary solutions of system (1). Denote $\bar{x}_t = x_t' - x_t''$, $\bar{\sigma}_t = \sigma_t' - \sigma_t''$, $\varphi_t = \varphi(t, \sigma_t') - \varphi(t, \sigma_t'')$. From (1) we have $\bar{x}_{t+1} = P\bar{x}_t + q\varphi_t$, and for the function $\bar{V}(\bar{x}) = \bar{x}^* H \bar{x}$, where $H = H^*$, analogously to the above, we obtain $\Delta\bar{V} \leqslant -\varepsilon|\bar{x}|^2$, $\varepsilon > 0$. Moreover, there exist numbers $\chi_2 \geqslant \chi_1 > 0$ such that

* $E\{(*)\}$ is the set of all x satisfying condition $(*)$.

** That is, $z^* \dfrac{\partial^2 V}{\partial x^2} z \geqslant 0$ for any real $\nu \times 1$-vectors x and z.

$$\varkappa_1^2|\bar{x}|^2 \leq V(\bar{x}) \leq \varkappa_2^2|\bar{x}|^2.$$

Let $\rho_0^2 = 1 - \varepsilon\varkappa_2^{-2} < 1$, $\varkappa_0 = \varkappa_2\varkappa_1^{-1}$. Then

$$V(\bar{x}_{t+1}) - \rho_0^2 V(\bar{x}_t) \leq 0.$$

For arbitrary t_0, $\bar{x}_{t_0}$ we obtain

$$V(\bar{x}_t) \leq V(\bar{x}_{t_0})\rho_0^{2(t-t_0)}, \qquad |x_t' - x_t''| \leq \varkappa_0\rho_0^{(t-t_0)}|x_{t_0}' - x_{t_0}''|,$$

$0 < \rho_0 < 1$, $0 < \varkappa_0 < \infty$. From the last inequalities and from the existence of the solution $x_0(t)$ of system (1) indicated in Theorem 1, the assertions of Theorem 2 follow.

The proof of Theorems 3 and 4 is carried out according to the same scheme as for Theorem 1, using the function

$$V(x) = x^*(H+H_0)x + 2\xi r^*(I-P^{-1})x\varphi(\sigma) + \xi\varkappa_0\varphi(\sigma)^2 + 2\vartheta\int_0^\sigma \varphi(\sigma')\,d\sigma', \qquad \sigma = r^* x,$$

introduced in (2), and certain techniques from (2).

The proof of Theorem 5 can be carried out by reducing the problem to the noncritical case (2) with the aid of the substitution

$$\varphi(\sigma) = \mu_0\sigma - \varphi_1(\sigma),$$

as is done, for example, in (7).

Received
6 II 1965

CITED LITERATURE

[1] Ya. Z. Tsypkin, DAN, 152, No. 2 (1963).
[2] Yu. A. Dmitriev, DAN, 160, No. 3 (1965).
[3] V. A. Yakubovich, DAN, 160, No. 2 (1965).
[4] G. P. Szegö, Proc. Nat. Akad. Sci., 50, No. 3 (1963).
[5] V. A. Yakubovich, Automation and Telemechanics, 25, No. 7 (1964).
[6] L. V. Kantorovich, G. P. Akilov, *Functional Analysis in Normed Spaces*, Moscow, 1959.
[7] M. A. Aizerman, F. R. Gantmacher, *Absolute Stability of Nonlinear Controlled Systems*, Publishing House of the USSR Academy of Sciences, 1963.

Note: Figure translations are in progress. See original paper for figures.