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Abstract

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ON THE UNIQUENESS OF DETERMINING THE SHAPE OF A BODY FROM THE VALUES OF THE EXTERNAL POTENTIAL

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1°. The following problem is considered in the article:

Let T_α $(\alpha = 1, 2)$ be connected domains (or open sets); let $\mu(y)$ be a function defined in the whole space E_n; and let $V_{T_\alpha}(x)$ be the potential determined by the body T_α with the given density $\mu(y)$. It is required to determine the mutual position of the domains T_α under the condition that the external potentials are equal, i.e.

$$V_{T_1}(x) = V_{T_2}(x) \quad \text{for } x \in E_n \setminus (\overline{T}_1 \cup \overline{T}_2). \tag{1}$$

This so-called inverse problem of potential theory belongs to the ill-posed problems of mathematical physics ([3]), since, in the formulation indicated, this problem, generally speaking, does not have a unique solution. But in applications of geophysics (see, for example, ([3], [11])) an important role is played by the question of a correct formulation of the problem under study, where one of the principal points is the proof of uniqueness of the solution of the problem.

For the case when the set $E_n \setminus (\overline{T}_1 \cup \overline{T}_2)$ has one component, under various restrictions on the domains T_α, for constant density the problem was considered in the papers ([1], [3], [4], [6–10]). The question of uniqueness of the solution of the problem with variable density for the logarithmic potential was studied by methods of the theory of functions of a complex variable in the papers ([6], [9]). For certain domains T_α, for which the set $E_n \setminus (\overline{T}_1 \cup \overline{T}_2)$ consists of more than one component, the problem in the formulation given above was considered in the papers ([8], [9]). In a somewhat different formulation the problem was considered in the paper ([11]), where examples of non-uniqueness of the solution of the problem are indicated. We note that in our formulation, for these examples condition (1) is not fulfilled, since the potentials $V_{T_1}(x)$ and $V_{T_2}(x)$ of the examples in ([11]) do not coincide identically at least in one of the components of the set $E_n \setminus (\overline{T}_1 \cup \overline{T}_2)$.

Below we give theorems on the uniqueness of the solution of the problem in the indicated formulation for the case of constant and variable density under various assumptions concerning the domains T_α of the space E_n $(n \geq 2)$. The proof is carried out for Newtonian and metaharmonic potentials ([5]); moreover, for the Newtonian potential, the theorems known earlier in the papers ([4–8]) are obtained as corollaries of the theorems proved. In the proof some ideas of the papers ([3–10]) are used, as well as a number of assertions of the paper ([2]).

2°. Denote by

$$V_{T_\alpha}(x) = \int_{T_\alpha} \mu(y) K(x, y)\, dy \quad (\alpha = 1, 2)$$

a potential with density $\mu(y)$. Here $K(x, y)$ is the fundamental solution of the metaharmonic equation

$$\Delta u - \chi^2 u = 0 \qquad (\chi = \text{const} \geq 0),$$

where, in the case $\chi > 0$, the kernel $K(x, y)$ is required to have the corresponding decay at infinity. In what follows we denote by $x = (x_1, \dots, x_n)$ a point of the space E_n; by dy, an element of volume; by $d_y S$, an element of the surface S_α; and by $\mathbf{n}_y$, the unit vector of the exterior normal to the surface S_α at the point y.

Let T_α $(\alpha = 1, 2)$ be finite connected domains bounded by the surfaces S_α, where S_α is the boundary of $E_n \setminus \overline{T}_\alpha$. Denote by S^e the boundary of the set $\overline{T}_1 \cup \overline{T}_2$. We introduce the notation (if $T_1 \neq T_2$):

$$S_1^i = S_1 \cap (\overline{T}_1 \cap \overline{T}_2), \quad S_1^e = S_1 \setminus S_1^i, \quad S_2^e = S_2 \cap S^e, \quad S_2^i = S_2 \setminus S_2^e.$$

If $T_1 = T_2$, then we set $S_\alpha^e = S_\alpha (\alpha = 1, 2)$. We note that some of the sets S_α^e, S_α^i may be empty. These notations coincide with those in the author’ s paper ([5]), where the case of simply connected domains was considered, and moreover

$$T_1 \cup T_2 \cup S^e = \overline{T}_1 \cup \overline{T}_2.$$

In addition, for simplicity in the formulation of the theorems, we shall assume that each boundary S_α belongs to the class $A^{(1,\lambda)}$ (the functions in a parametric representation of the boundary are differentiable and their first derivatives satisfy a Hölder condition with exponent λ, $0 < \lambda < 1$).

3°. Denote by (ρ, θ) the spherical coordinates of the point x of the space E_n, and by $(\mathbf{R}, \mathbf{n}_y)$ the scalar product of the vector $\mathbf{R}$ with the vector $\mathbf{n}_y$.

Theorem 1 (the case $\chi = 0$). *If there exists at least one point $O \in (\overline{T}_1 \cap \overline{T}_2)$ such that:*

1) for the radius vector $\mathbf{R}_y = \overrightarrow{Oy}$ (with origin at the point O) and the unit vector $\mathbf{n}_y$ of the exterior normal to the surface S_α at the point y, the relation holds

$$(\mathbf{R}_y, \mathbf{n}_y) \geq 0 \qquad \text{for} \qquad y \in S_1^i, S_2^i;$$

2) for the positive function $\mu(y) \in C^{(1)}$ in $(\overline{T}_1 \cup \overline{T}_2)$ the condition is fulfilled

$$\frac{\partial}{\partial \rho}(\rho^n \mu) > 0, \qquad \rho \neq 0, \qquad y \in (\overline{T}_1 \cup \overline{T}_2) \quad (n \geq 2);$$

3) for the domains T_α, with the given density, the equality holds

$$V_{T_1}(x) = V_{T_2}(x) \qquad \text{for } x \in E_n \setminus (\overline{T}_1 \cup \overline{T}_2),$$

then

$$T_1 = T_2.$$

Condition 1) of Theorem 1 means that each of the sets S_α^e $(\alpha = 1, 2)$ is nonempty, and the set $\overline{T}_1 \cap \overline{T}_2$ is star-shaped with respect to the point O, i.e., every ray with center at the point O intersects S_α^i $(\alpha = 1, 2)$ either at one point or along a rectilinear segment.

For the case $(n = 2)$ of a logarithmic potential with constant density, we shall also give a theorem for the domains constructed below.

Let $\widehat{G}$ be an arbitrary bounded simply connected domain of the plane $w = \xi_1 + i\xi_2$, and let $f(z)$ $(z = \rho e^{i\theta})$, $f(0) = 0$, $f'(0) = 1$, be a function holomorphic and univalent in the disk $|z| < R$, mapping the disk $|z| < R$ (R an arbitrary positive number) onto the domain $\widehat{G}$ of the w-plane. Denote by K_{ρ_0} the disk $|z| < \rho_0$, where $\rho_0 = \text{const}$ and $\rho_0 \leq (2 - \sqrt{3})R$. Let T_α $(\alpha = 1, 2)$ be connected bounded domains $T_\alpha \subset K_{\rho_0}$ such that there exists a point $O \in (\overline{T}_1 \cap \overline{T}_2)$ and $(\mathbf{R}_y, \mathbf{n}_y) \geq 0$ for $y \in S_1^i, S_2^i$.

Denote by G_α the connected domains of the w-plane that are the images of the domains T_α under the mapping by the function $f(z)$.

Theorem 2. *If, under the assumptions made, in the domains G_α $(\alpha = 1, 2)$ the equality*

$$\int_{G_1} \ln \frac{1}{r_{\xi\eta}}\, d\eta = \int_{G_2} \ln \frac{1}{r_{\xi\eta}}\, d\eta \quad \text{for points } \eta \in E_2 \setminus (\overline{G}_1 \cup \overline{G}_2),$$

holds, then

$$G_1 = G_2.$$

We note that, for the logarithmic potential, under the assumption that each domain T_α is star-shaped with respect to a common interior point, Theorems 1 and 2 were proved in [6].

Theorem 3 (the case $\varkappa \geq 0$). *If there exists at least one constant vector* $\mathbf{q} \neq 0$, $\mathbf{q} = (q_1, \dots, q_n)$, *such that:*

1) *each of the sets* S^e_α $(\alpha = 1, 2)$ *is nonempty, and a line parallel to the vector* $\mathbf{q}$ *intersects* $S^i_1 \cup S^i_2$ *in no more than two points or two segments;*
2) *for the positive function* $\mu \in C^{(1)}$ *in* $(\overline{T}_1 \cup \overline{T}_2)$, *the condition* $\partial\mu/\partial y_k = 0$ *is satisfied (the direction of the axis* y_k *coincides with the vector* $\mathbf{q}$*);*
3) *for the domains* T_α, *with the given density, the equality*

$$V_{T_1}(x) = V_{T_2}(x) \quad \text{for } x \in E_n \setminus (\overline{T}_1 \cup \overline{T}_2),$$

holds, then

$$T_1 = T_2.$$

Let γ, β be two arbitrary numbers, with $\gamma^2 + \beta^2 \neq 0$, and let $\mathbf{q} = (q_1, \dots, q_n)$ be a constant vector.

Theorem 4 (the case $\varkappa = 0$). *If there exist a point* O, *numbers* γ, β, *and a vector* $\mathbf{q}$ *such that:*

1) *for the surfaces* S_α *the relation*

$$\int_{S^i_1} |\Phi(y)|\, d_y S + \int_{S^i_2} |\Phi(y)|\, d_y S < \int_{S^e_1} |\Phi(y)|\, d_y S + \int_{S^e_2} |\Phi(y)|\, d_y S,$$

holds, where

$$\Phi(y) = (\gamma \mathbf{R}_y + \beta \mathbf{q}, \mathbf{n}_y);$$

2) *for the domains* T_α, *with density* $\mu = 1$, *the equality*

$$V_{T_1}(x) V_{T_2}(x), \quad \text{for } x \in E_n \setminus (\overline{T}_1 \cup \overline{T}_2),$$

holds, then

$$T_1 = T_2.$$

If $\gamma = 1$ and $\beta = 0$, then $\Phi(y) = (\mathbf{R}_y, \mathbf{n}_y)$, and condition 1) of Theorem 4 is replaced by the condition

$$\int_{S_1^i} |(\mathbf{R}_y, \mathbf{n}_y)|\, d_y S + \int_{S_2^i} |(\mathbf{R}_y, \mathbf{n}_y)|\, d_y S < \int_{S_1^e} |(\mathbf{R}_y, \mathbf{n}_y)|\, d_y S + \int_{S_2^e} |(\mathbf{R}_y, \mathbf{n}_y)|\, d_y S, \tag{1'}$$

which refines the formulation of Theorem 1 of [8].

We also note that the assertion of Theorem 4 remains valid if, instead of condition $(1')$, one requires the fulfillment of the condition

$$\int_{S_1^i} |(\mathbf{R}_y, \mathbf{n}_y)|\, d_y S + \int_{S_2^i} |(\mathbf{R}_y, \mathbf{n}_y)|\, d_y S < \int_{S_1^e} (\mathbf{R}_y, \mathbf{n}_y)\, d_y S + \int_{S_2^e} (\mathbf{R}_y, \mathbf{n}_y)\, d_y S. \tag{1''}$$

It is obvious that when the right-hand side of $(1')$ is not equal to infinity, $(1'')$ implies $(1')$. Let us note that condition $(1'')$ is satisfied, in particular, when $(\overline{T}_1 \cap \overline{T}_2)$ is star-shaped with respect to some point $O \in (\overline{T}_1 \cap \overline{T}_2)$.

If $\gamma = 0$ and $\beta = 1$, then $\Phi(y) = (\mathbf{q}, \mathbf{n}_y)$, and condition 1) of Theorem 4 is replaced by the condition

$$\int_{S_1^i} |(\mathbf{q}, \mathbf{n}_y)|\, d_{yS} + \int_{S_2^i} |(\mathbf{q}, \mathbf{n}_y)|\, d_{yS} < \int_{S_1^e} |(\mathbf{q}, \mathbf{n}_y)|\, d_{yS} + \int_{S_2^e} |(\mathbf{q}, \mathbf{n}_y)|\, d_{yS}. \tag{1'''}$$

We note that if condition 1) of Theorem 4 is replaced by condition $(1''')$, then the theorem is valid for the case $\chi \geqslant 0$. In this case the theorem refines the author' s theorem ([5]) for $\chi > 0$ and Theorem 2 of ([8]) for $\chi = 0$. In particular, if a line parallel to the vector $\mathbf{q}$ intersects $S_1^i \cup S_2^i$ in no more than two points or segments, then condition $(1''')$ is satisfied. Therefore it implies the assertion of the theorem of the paper ([7]) without the additional assumption that the center of mass lies inside the body.

Remark. The smoothness condition on the boundaries S_α $(\alpha = 1, 2)$ can be substantially weakened in the formulation of Theorem 3 (see ([1,10])). This assertion also holds for Theorems 1 and 2 if, in the formulations of Theorems 1 and 2, the condition $(\mathbf{R}_y, \mathbf{n}_y) \geqslant 0$ on S_α^i $(\alpha = 1, 2)$ is replaced by the geometric requirement: each of the sets S_α^e is nonempty, and $\overline{T}_1 \cap \overline{T}_2$ is star-shaped with respect to the point $O \in (\overline{T}_1 \cap \overline{T}_2)$. An analogous conclusion about the smoothness of S_α can be drawn for Theorem 4, if conditions $(1')$–$(1''')$ are given some geometric meaning.

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Note: Figure translations are in progress. See original paper for figures.