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Abstract

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HYDROMECHANICS

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ON THE INFLUENCE OF ROTATION ON THE TURBULIZATION OF PLANE FLOWS

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In papers (^{1}, ^{2}) it was shown that rotation imparts to a fluid an effective gyroscopic elasticity with respect to disturbances involving deformation of the force tubes of the angular-velocity vector $\vec{\Omega}$. At the same time, in two-dimensional flows, when the velocity $\mathbf{V}$ is perpendicular to $\vec{\Omega}$ and $(\vec{\Omega}\nabla)\mathbf{V} = 0$, the Coriolis forces are potential and lead only to a redistribution of pressure. If phase transitions are not taken into account, the motion of a fluid of this type does not depend on the general rotation of the system.

There are grounds for believing (^{2}) that the turbulence of two-dimensional flows in a rapidly rotating fluid must also be two-dimensional, since instabilities with respect to three-dimensional disturbances should be suppressed by gyroscopic elasticity. The principal mechanism of turbulization at sufficiently large Reynolds numbers Re will then be instability near inflection points of the velocity profile (^{3}). In the absence of rotation such flows are unstable both with respect to two-dimensional and to three-dimensional disturbances. In the present paper, using the example of a tangential discontinuity, it will be shown that sufficiently rapid rotation stabilizes a three-dimensional disturbance.

Let us direct the z-axis along $\vec{\Omega}$, the x-axis along the normal to the surface of discontinuity (which we take to be plane), and the y-axis along the flow. The motion of an incompressible fluid in a rotating coordinate system is described by the equations

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\mathbf{V} = -\frac{1}{\rho}\nabla p^* + 2[\mathbf{V}\vec{\Omega}], \qquad \operatorname{div}\mathbf{V} = 0, \tag{1}$$

where $p^* = p + \rho\varphi$; φ is the potential of the centrifugal forces; $2[\mathbf{V}\vec{\Omega}]$ is the density of the Coriolis forces. In the case of a tangential discontinuity, the equations for small pulsations of the velocity $\mathbf{v}$ and pressure p have the form

$$\frac{\partial \mathbf{v}_q}{\partial t} + (\mathbf{V}_{0q}\nabla)\mathbf{v}_q = -\frac{1}{\rho}\nabla p_q + 2[\mathbf{v}_q\vec{\Omega}], \qquad \operatorname{div}\mathbf{v}_q = 0, \tag{2}$$

where the indices $q = 1$ and $q = 2$ correspond to the regions $x > 0$ and $x < 0$. Without loss of generality one may assume that $V_{01} = V_0$, and $V_{02} = -V_0$.

At the boundary the normal displacements and the momentum flux must be continuous. In the absence of rotation the latter condition reduces to the requirement of pressure continuity (([4]), §30); although in the case $\Omega \neq 0$ the momentum-flux tensor Π_{ik} is substantially modified, the boundary conditions remain the same. To obtain Π_{ik} for $\Omega \neq 0$, one should in the expression for Π_{ik} at $\Omega = 0$ (see ([4]), §7) replace p by p^* and add the tensor $\widetilde{\Pi}_{ik}$, which takes into account the Coriolis force. To determine $\widetilde{\Pi}_{ik}$, introduce the vector potential of the velocity $\mathbf{V} = \operatorname{rot} \mathbf{A}$. We have

$$\mathbf{f} = 2\rho[\mathbf{V}\vec{\Omega}] = -2\rho[\vec{\Omega} \operatorname{rot} \mathbf{A}] = -2\rho\{\nabla(\mathbf{A}\vec{\Omega}) - (\vec{\Omega}\nabla)\mathbf{A}\},$$

or, in tensor notation,

$$f_i = -\frac{\partial}{\partial x_k} 2\rho\{A_l \Omega_l \delta_{ik} - \Omega_k A_i\} = -\frac{\partial}{\partial x_k} \widetilde{\Pi}_{ik}. \tag{3}$$

To obtain from $\widetilde{\Pi}^*_{ik}$ the tensor $\widetilde{\Pi}_{ik}$, one must symmetrize $\widetilde{\Pi}^*_{ik}$, i.e., subtract the tensor $2\rho\Omega_i A_k$. In this case the Euler equation $\rho\, \partial V_i/\partial t = -\partial \Pi_{ik}/\partial x_k$ must not change. Hence we find the gauge condition for $\mathbf{A}$: $\partial \Omega_i A_k/\partial x_k = 0$, or

$$\operatorname{div} \mathbf{A} = 0. \tag{4}$$

Thus,

$$\widetilde{\Pi}_{ik} = 2\rho\left(A_l \Omega_l \delta_{ik} - A_i \Omega_k - A_k \Omega_i\right), \tag{5}$$

where $\mathbf{A}$ satisfies (4). Since the velocities $\mathbf{V}$ are assumed bounded, the components of $\mathbf{A}$ must be continuous. Therefore, also for $\Omega \neq 0$, the discontinuity of Π_{ik} reduces to the discontinuity of the pressure.

As noted above, for two-dimensional disturbances rotation plays no role, and therefore the results obtained for the case $\Omega = 0$ ([4], § 30) are automatically transferred to the problem under consideration: the discontinuity proves to be unstable with respect to two-dimensional disturbances ($k_z = 0$) for any values of the wave vector $k = k_y$, and the increment is equal to kV_0.

In the case of three-dimensional disturbances ($k_z \neq 0$), eliminating from (2) all variables except p_q, we obtain the following equations for p_q (the solution is sought in the form $f(x)e^{i(k_y y + k_z z - \omega t)}$):

$$\frac{d^2 p_q(x)}{dx^2} - \varkappa_q^2 p_q(x) = 0, \quad \text{where} \quad \varkappa_q^2 = k^2 - \frac{4\Omega^2 k_z^2}{(k_y V_{0q} - \omega)^2}, \quad k^2 = k_y^2 + k_z^2. \tag{6}$$

For $\varkappa_q^2 < 0$ the solution is not localized near the plane of discontinuity and therefore, on energetic grounds, cannot be unstable. For $\varkappa_q^2 < 0$ one should set $p_1(x) \sim e^{-\varkappa_1 x}$, $p_2(x) \sim e^{\varkappa_2 x}$, so that as $x \to \pm\infty$, $p \to 0$. From (2) we obtain the relations between p_q and v_{xq}:

$$v_{xq} = \frac{i}{\rho} \frac{2\Omega k_y p_q(x) + (k_y V_{0q} - \omega)\, dp_q(x)/dx}{(k_y V_{0q} - \omega)^2 - 4\Omega^2}. \tag{7}$$

The displacement of the boundary along the normal ξ is related to v_{xq} by the relations

$$v_{xq}(\xi) = \frac{d\xi}{dt} = \frac{\partial \xi}{\partial t} + V_{0q}\frac{\partial \xi}{\partial y} = i(k_y V_{0q} - \omega)\xi. \tag{8}$$

Expressing from (7) $v_{xq}(\xi)$ in terms of ξ, substituting into (7), and equating $p_1(\xi)$ and $p_2(\xi)$, we obtain the following characteristic equation for ω:

$$\frac{(k_y V_0 + \omega)\,[2k_y\Omega - (k_y V_0 - \omega)\varkappa_1]}{4\Omega^2 - (k_y V_0 - \omega)^2} + \frac{(k_y V_0 - \omega)\,[2k_y\Omega - (k_y V_0 + \omega)\varkappa_2]}{4\Omega^2 - (k_y V_0 + \omega)^2} = 0. \tag{9}$$

It is easy to see that (9) has the form $F(\omega^2, k^2, V_0, \Omega) = 0$ and contains ω, k_y, and k_z only in even powers.

Let us determine the region of aperiodic instability. To this end, put $\omega = 0$ in (9). It can be shown that the case $k_y \to 0$ corresponds to the real frequency $\omega \to -2\Omega$. Therefore the boundary of the instability region is determined by the equation

$$2\Omega = V_0\sqrt{k^2 - \frac{4k_z^2\Omega^2}{k_y^2 V_0^2}}. \tag{10}$$

Putting $k_y = k\sin\theta$, $k_z = k\cos\theta$, we obtain that on the boundary of the instability region

$$k_{y0} = 2\Omega/V_0. \tag{11}$$

If the fluid is bounded by the planes $z = \pm L$, then from the condition $v_z|_{z=\pm L} = 0$ it follows that $k_{zn} = \pi n/2L \geq \pi/2L$ $(n = 1, 2, ...)$. The case $n = 0$ describes the two-dimensional disturbances considered above. Consequently, the minimum angle θ_0 at which $\omega = 0$ is determined by the equation

$$\operatorname{tg}\theta_0 = 4L\Omega/\pi V_0, \tag{12}$$

and in the case of rapid rotation ($a = 4L\Omega/\pi V_0 \gg 1$), $\theta_0 \to \pi/2$.

This result agrees well with the conclusion (^{1}) concerning the reduction of the effect of gyroscopic elasticity with decreasing wavelength of the disturbance in the direction perpendicular to $\vec{\Omega}$.

Expanding (9) in a series up to and including terms $\sim \omega^2$, one can show that the region of instability corresponds to the angles $\pi/2 > \theta > \theta_0$ and rapidly shrinks as the parameter a increases. For $\sigma = \pi/2 - \theta \ll 1/a$, the instability increment is

$$\gamma \approx 2\Omega/a\sigma \approx \pi V_0/2\lambda_y, \tag{13}$$

where λ_y is the wavelength along the y-axis. If one takes into account the small but finite viscosity of the fluid ν, and notes that in the instability region its development time is $\tau = 1/\gamma$ (as $\theta \to \theta_0$, $\tau \to \infty$), then it is easy to see that, for sufficiently large Ω, three-dimensional disturbances can be completely suppressed. For this it is necessary that the time of viscous dissipation $\tau_\nu \approx \lambda_y^2/\nu$ be less than the growth time τ. Hence, from (11), we obtain the following criterion for suppression of three-dimensional turbulence:

$$\nu\Omega > aV_0^2, \tag{14}$$

where a is a certain constant ~ 1.

Thus, we have shown that, for sufficiently large Ω, the development of instability of a tangential discontinuity in a rotating fluid leads to the onset of two-dimensional turbulence. We note that nonlinear effects lead to the establishment of a spectrum of two-dimensional pulsations corresponding to the Kolmogorov-Obukhov law ($v_\lambda \sim \lambda^{1/3}$). The generation of three-dimensional disturbances due to nonlinearity is strongly impeded, since in the case of three-dimensional waves ω decreases with increasing k (^{1}). As is known, in this case the interaction of waves is sharply weakened (in particular, for an analogous reason, at low temperatures liquid helium becomes superfluid (^{5})). Undoubtedly, the results obtained remain valid also in the case of slightly smeared tangential discontinuities (i.e., near inflection points of the velocity profile).

In conclusion, we note that the occurrence of two-dimensional pulsations under conditions close to those considered was observed experimentally in the experiments of Hide and Fultz (^{6}) on thermal convection in a rotating fluid. The authors called the observed phenomenon vacillations. It follows from the results obtained that the two-dimensional character of turbulence is a general regularity of the hydrodynamics of a rapidly rotating fluid and is due to the effect of gyroscopic elasticity.

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Note: Figure translations are in progress. See original paper for figures.

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