ON THE HEIGHT OF SIMPLE LIE ALGEBRAS

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Abstract

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MATHEMATICS

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1. Let Ω be a simple finite-dimensional Lie algebra over an algebraically closed field Φ; let x be a nonzero element of Ω. Define on Ω an increasing x-filtration $(\Omega^{\langle i \rangle})$, where $\Omega^{\langle i \rangle}$ is the subspace spanned by all possible products of the form $[\cdots[[xy_1]y_2]\cdots y_k]$, $y_1, \ldots, y_k \in \Omega$, $0 \leq k \leq i$; $\Omega^{\langle -1 \rangle} = 0$. Obviously, $\Omega^{\langle 0 \rangle} = \{x\}$, $\Omega^{\langle i \rangle} \subseteq \Omega^{\langle i+1 \rangle}$, and $[\Omega^{\langle i \rangle}, \Omega^{\langle j \rangle}] \subseteq \Omega^{\langle \min(i,j)+1 \rangle} \subseteq \Omega^{\langle i+j \rangle}$.

The minimal index n for which $\Omega^{\langle n \rangle} = \Omega$ will be called the **height** $\Delta(\Omega, x)$ of the algebra Ω relative to the element x. The existence of such an index is ensured by the finite-dimensionality and simplicity of the algebra Ω. In other words, $\Delta(\Omega, x) + 1$ is the number of nonzero homogeneous components $\overline{\Omega}^{\langle i \rangle}$ of the graded Φ-module associated in the usual way with the algebra Ω with the x-filtration. The multiplicative structure of this Φ-module, incidentally, is trivial. The integer

$$\Delta(\Omega) = \max_{x \in \Omega} \Delta(\Omega, x)$$

will be called the **height of the simple algebra** Ω.

The concept of height eliminates the need for a lengthy verbal description of various situations arising in degenerate simple p-Lie algebras. Let us first note the following, directly verifiable

Proposition 1. *There exists an absolute constant* ρ *such that* $\Delta(\Omega) \leq \rho$ *for every simple Lie algebra* Ω *over a field* Φ *of characteristic zero.*

The same is true for algebras classical in the sense of Seligman-Mills (1). On the other hand, the Witt algebra $\mathfrak{W}_1$ of dimension p has height $p - 1$ (i.e., it may be arbitrarily large), while the height of the Witt-Jacobson algebra $\mathfrak{W}_n$ (2) grows without bound together with n for fixed p. Thus the degeneracy of the simple algebra Ω has a decisive effect on the behavior of its height.

2. From the facts known about Lie algebras with strong degeneracy (see (3,4)) it follows directly that their height $\Delta \geq p - 2$. More precisely, $\Delta(\Omega, c) \geq p - 2$ for every element $c \in \Omega^{(\frac{p-3}{2})}$, defined by the condition

$$cX^{ic} = 0, \qquad i = 0, 1, 2, \ldots, p - 4. \tag{1}$$

The notation is taken from the paper (^{3}), where, incidentally, it was noted (in somewhat different terms) that if $\Delta(\Omega, c) = p-2$, then the dimension of the algebra Ω must be sufficiently large. In fact, the following is true.

Theorem 1. *In a simple Lie algebra Ω with strong degeneracy, the inequality*

$$\Delta(\Omega, c) \geq p-1$$

holds for every nonzero element $c \in \Omega^{(\frac{p-3}{2})}$.

It is convenient to prove this by contradiction. Denote by $\mathfrak{M}$ the subspace in Ω spanned by all elements of the form $[cx^{p-2}c]$, $x \in \Omega$. Since, by assumption, $\Delta(\Omega, c) = p-2$, it follows that $[ca] \in \mathfrak{M}$ for every ele-

element $a \in \mathfrak{L}$. In this case

$$[cx^{p-2}c] = [[cx] \cdot x \dots xc] \in [\mathfrak{M}x^{p-3}c].$$

Using this observation a sufficient number of times, we express the product $[ca]$ as a linear combination of elements

$$[ca_0^{p-2}ca_1^{p-3}ca_2^{p-3}c \dots ca_m^{p-3}c], \tag{2}$$

where m is a prescribed positive integer. Lemma 1, formulated below (which is also of independent interest), shows that elements of type (2) with large indices m are equal to zero. Consequently, $[ca] = 0$ or c is an element of the center of the algebra $\mathfrak{L}$, which is impossible.

Lemma 1. *The subspace $\mathfrak{M}$ is invariant under the endomorphisms $X^{p-3}C$ of the vector space $\mathfrak{L}$. The associative algebra generated by all endomorphisms $\sigma_x = X^{p-3}C|_{\mathfrak{M}}$, $x \in \mathfrak{L}$, is nilpotent.*

The first assertion of the lemma is a simple consequence of identity (*) in paper (5). From the identities given in § 1 of the same paper there follow the following three properties of the endomorphisms σ_x: 1) $\sigma_x^2 = 0$; 2) $\sigma_x\sigma_y\sigma_x = 0$; 3) the vector space $\mathfrak{S}$, spanned by all endomorphisms σ_x, is closed under the operation $\circ$: $\sigma_x \circ \sigma_y = \sigma_x\sigma_y - \sigma_y\sigma_x$. Thus $\mathfrak{S}$ is a Lie algebra of linear transformations acting on $\mathfrak{M}$. In view of properties 1) and 2), the elements σ_x have nilpotency index two in $\mathfrak{S}$: $((\mathfrak{S} \circ \sigma_x) \circ \sigma_x) = 0$. All such elements form a weakly closed system (see (6), Ch. II, § 1, 2). The enveloping algebra of this system coincides with the enveloping algebra $\overline{\mathfrak{S}}$ of the algebra $\mathfrak{S}$. Its nilpotency is ensured by Jacobson’ s theorem.

3. Proposition 2. *In a simple Lie algebra $\mathfrak{L}$ with a strong degeneration there exists a nonzero element $c_0 \in \mathfrak{L}^{(\frac{p-3}{2})}$ whose associated endomorphism C_0 satisfies the identity*

$$C_0X^{p-3}C_0Y^{p-3}C_0 = 0, \qquad x, y \in \mathfrak{L}. \tag{3}$$

This assertion, in view of Lemma 1, is proved almost in the same way as Lemma 3.2 in paper (5). The property of the element c_0 expressed by identity (3) is very important for the description of simple p-Lie algebras with relative height $\Delta(\mathfrak{L}, c_0) = p - 1$. Besides $\mathfrak{W}_1$, there are less trivial examples of simple p-Lie algebras in which all these conditions are satisfied (one of the most interesting examples is the Block algebra (7)). We note further that identity (3) is in a certain sense optimal. Indeed, let $c \in \mathfrak{L}^{\left(\frac{p-3}{2}\right)}$ and $\Delta(\mathfrak{L}, c) = p - 1$. The identity $CX^{p-3}C = 0$, for obvious reasons, cannot hold. On the other hand, in many algebras the subspace $\mathfrak{M}$ associated with the element c contains this element, and it is quite natural to consider the case when $\mathfrak{M} = \{c\}$. In other words, let

$$[cx^{p-2}c] = \Gamma(x) \cdot c, \qquad x \in \mathfrak{L}, \quad \Gamma(x) \in \Phi, \tag{4}$$

where $\Gamma(x) \neq 0$ for at least one element $a \in \mathfrak{L}$.

An element $c \in \mathfrak{L}^{\left(\frac{p-3}{2}\right)}$ satisfying condition (4) will be called *stable*. It is easy to see that the requirement of stability is less restrictive than the identity $CX^{p-3}C = 0$, but more stringent than (3).

Theorem 2. *A simple p-Lie algebra $\mathfrak{L}$ with a strong degeneration, distinct from the Witt algebra $\mathfrak{W}_1$, cannot have a stable element c with respect to which $\Delta(\mathfrak{L}, c) = p - 1$.*

The proof, based on the additive properties of the multiplier $\Gamma(x)$, is rather long. As an intermediate stage of the argument it is established that a simple p-Lie algebra $\mathfrak{T}$ with a stable element c, $\Delta(\mathfrak{L}, c) = p - 1$, must have a basis of the form c; $[ce_1], \dots, [ce_r]$; some-

the second part of the products $[ce_ie_j]$, $[ce_1^i]$, $\cdots$, $[ce_r^i]$; $r \geqslant 1$; $i = 3, \dots, p - 1$. Here $e_k = [ce_k^{p-2}]$, $[ce_k^{p-2}c] = 2c$, $1 \leqslant k \leqslant r$.

Without dwelling on the other details, let us show how the conclusion of the theorem already follows from this, if one additionally assumes that on $\mathfrak{L}$ there is defined some invariant bilinear form $f(x, y)$ (not connected with any representation). From the identities (1), which define the element c, it follows that $f(c, x) \neq 0 \Rightarrow f(c, [ce_k^{p-2}]) \neq 0$ or $f(c, e_k) \neq 0$ for some k. But

$$f(c, [ce_k^{p-2}]) = \frac{1}{2} f([[ce_k^{p-2}]c], [ce_k^{p-2}]) = 0$$

by the invariance of f. Further,

$$f(c, e_k) = \frac{1}{2} f([ce_k^{p-2}c], e_k) = -\frac{1}{2} f(c, [[ce_k^{p-2})e_k]) = -\frac{1}{2} f(c, e_k),$$

i.e. also $f(c, e_k) = 0$. The theorem in this case is proved. It is worth recording the lemma used in the construction, indicated above, of a basis of the algebra $\mathfrak{L}$.

Lemma 2. Let $\mathfrak{L}$ be an arbitrary p-algebra of Lie, $c \in \mathfrak{L}^{(p-3)/2}$, and let $[ca^{p-2}c] = c$ for some element $a \in \mathfrak{L}$. Put $e = [ca^{p-1}]$. Then $[ce^{p-2}c] = 2c$. If, in addition, c is a stable element, then $[ce^{p-1}] = e$.

Proof. First,

$$[cx^{p-1}c] = -\frac{1}{2}[cx^{p-2}cx]$$

for every $x \in \mathfrak{L}$. Therefore

$$[ce] = -[ca^{p-1}c] = \frac{1}{2}[ca^{p-2}ca] = \frac{1}{2}[ca].$$

Suppose that the equality $[ce^k] = (1/2)^k[ca^k]$ has already been proved. Applying the known relations (see (6), p. 49, formula (6)), we find:

$$[ce^{k+1}] = (1/2)^k[ca^k e] = (1/2)^k \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}[c[ea^{k-i}]a^i].$$

But $[ea] = [ca^p]$. Since $\mathfrak{L}$ is a p-algebra, $[c[ea^{k-i}]] = -[ea^{k-i}c] = -[ca^p \cdot a^{k-i-1}c] = 0$ for $k-i < p-2$ (see (1)). If $k < p-2$, then also $k-i < p-2$, and then

$$[ce^{k+1}] = (1/2)^k[cea^k] = (1/2)^k(1/2)[ca \cdot a^k] = (1/2)^{k+1}[ca^{k+1}].$$

In particular, $[ce^{p-2}] = 2[ca^{p-2}]$ and $[ce^{p-2}c] = 2c$. For $k = p-2$ one obtains a somewhat different expression:

$$[ce^{p-1}] = (1/2)^{p-2}[ca^{p-2}e] = (1/2)^{p-2}[cea^{p-2}] - (1/2)^{p-2}[c[ea^{p-2}]] = [ca^{p-1}] + 2[ca^p \cdot a^{p-3}c].$$

In view of the fact that now c is a stable element, we have

$$e = [ce^{p-1}] + \varepsilon \cdot c. \tag{5}$$

Using (5) and the expression for $\Lambda(a,b)$ in the formula

$$(a+b)^p = a^p + b^p + \Lambda(a,b),$$

we find

$$e^p = ([ce^{p-1}] + \varepsilon c)^p = [ce^{p-1}]^p + \varepsilon[c[ce^{p-1}]^{p-1}] + \varepsilon^2[c[ce^{p-1}]^{p-2}c]$$

$$= [ce^{p-1}]^p + \varepsilon[c(e-\varepsilon c)^{p-1}] + \varepsilon^2[c(e-\varepsilon c)^{p-2}c] = [ce^{p-1}]^p + \varepsilon([ce^{p-1}] - \varepsilon[ce^{p-2}c]) + \varepsilon^2[ce^{p-2}c],$$

or

$$e^p = [ce^{p-1}]^p + \varepsilon[ce^{p-1}].$$

Hence, also from relation (5),

$$(e-\varepsilon c)^p = [ce^{p-1}]^p = e^p - \varepsilon[ce^{p-1}].$$

But, on the other hand, direct computations give

$$(e-\varepsilon c)^p = e^p - \varepsilon[ce^{p-1}] + \varepsilon^2[ce^{p-2}c].$$

Comparing the expressions obtained, we find $\varepsilon^2[ce^{p-2}c] = 0$, i.e. $\varepsilon = 0$. Thus relation (5) coincides with what is asserted in the lemma.

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