APPROXIMATION THEOREMS FOR DIFFERENTIAL EQUATIONS

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Abstract

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E. M. SAK

APPROXIMATION THEOREMS FOR DIFFERENTIAL EQUATIONS

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This paper investigates the possibility of uniform approximation by polynomial solutions of the polyanalytic equation

$$D^m u = 0,$$

where m is a natural number, $D \equiv D_z \equiv \partial/\partial x + i\,\partial/\partial y$ is the complex derivative, D^m is its m-th iteration, $z = x + iy$, and also of the more general equation

$$D^k \overline{D}^l u = 0, \tag{1}$$

where k, l are (fixed) natural numbers, $\overline{D} \equiv \partial/\partial x - i\,\partial/\partial y$ is the complex conjugate derivative, $\overline{D}^l$ is its l-th iteration, and the rate of approximation by harmonic polynomials.

Equation (1) contains, as a special case (for $k = l$), the polyharmonic equation $\Delta^k u = 0$, where Δ is the Laplace operator.

Sufficiently smooth solutions of equation (1) in a simply connected domain are characterized by the representation (see (1), p. 173)

$$u(z) = \sum_{j=0}^{k-1} \bar{z}^j \varphi_j(z) + \sum_{j=0}^{l-1} z^j \overline{\psi_j(z)} \tag{2}$$

in terms of single-valued analytic functions $\varphi_j(z), \psi_j(z)$ of the complex variable z.

From representation (2) there follows an analogue of Runge's theorem: a solution of equation (1) in a simply connected bounded domain can be expanded in a series of polynomial solutions of the same equation, converging uniformly inside the domain.

In studying the possibility of approximation on closed sets, the following lemma on approximation of the logarithmic kernel will be used essentially. We shall agree to assume that the bounded closed set E is situated in the disk $|z| < 1$, and throughout the article, in place of absolute positive constants, we shall write const.

Lemma. *Let the compact set E have connected complement, and let R_δ be the set of all points of E whose distance from its boundary is less than δ, $0 < \delta < 1$. For any $\zeta \in R_\delta$ and natural number n there exists a harmonic polynomial $h_\zeta(z)$ such that*

$$\left|\ln|\zeta - z| - h_\zeta(z)\right| < (\text{const})^n \min\left\{\frac{\delta^n}{|\zeta - z|^n}; \frac{\delta}{|\zeta - z|}\right\}, \quad z \in E,$$

and moreover

$$|h_\zeta(z)| < (\text{const})^n \ln\frac{1}{\delta}, \quad z \in E,$$

and the degree of $h_\zeta(z)$ is bounded by a number independent of ζ.

Proof. Suppose, for simplicity, that $\zeta = 0$. S. N. Mergelyan [2] in fact established the existence on E of an analytic function $w(z)$ satisfying the conditions:

1) $|z - w(z)| < \text{const} \cdot \delta,\ z \in E$;
2) $|w(z)| > \text{const} \cdot |z|,\ z \in E$;
3) $|w(z)| > \text{const} \cdot \delta,\ z \in E$.

The function

$$S_n(z) = \ln|w| + \operatorname{Re}\sum_{k=1}^{n-1}\frac{1}{k}\left(\frac{w - z}{w}\right)^k, \qquad w = w(z),$$

is harmonic on E, and

$$|\ln|z| - S_n(z)| = \left|\operatorname{Re}\sum_{k=n}^{\infty}\frac{1}{k}\left(\frac{w - z}{w}\right)^k\right| < (\text{const})^n\frac{\delta^n}{|z|^n},$$

$$z \in E, \quad |z| > \text{const} \cdot \delta,$$

$$|\ln|z| - S_n(z)| \le \left|\ln\left|\frac{z}{w}\right|\right| + \left|\sum_{k=1}^{n-1}\frac{1}{k}\left(\frac{w - z}{w}\right)^k\right| < (\text{const})^n\frac{\delta}{|z|}$$

for the remaining $z \in E$.

By Runge’ s theorem there exists a harmonic polynomial $h_0(z)$ such that

$$|S_n(z) - h_0(z)| < \delta^n, \quad z \in E.$$

From this and from the preceding estimates we conclude that

$$|\ln|z| - h_0(z)| < (\mathrm{const})^n \min\{\delta^n/|z|^n;\ \delta/|z|\}, \qquad z \in E.$$

It is not difficult to obtain also the estimate for $|h_0(z)|$ asserted in the lemma.

Finally, let us show that, by virtue of the continuity of the function $\ln|\zeta - z|$ for $\zeta \neq z$, one may restrict oneself to choosing, as $h_\zeta(z)$, a finite number (N) of harmonic polynomials $h_{\zeta_j}(z)$, $j = 1, 2, \dots, N$. This will complete the proof of the lemma.

For every $\delta > 0$ there is an $\varepsilon > 0$ such that

$$\big|\ln|\zeta - z| - \ln|z|\big| < \delta^n, \qquad |\zeta| < \varepsilon, \quad |z| > \delta^2, \quad z \in E.$$

But then

$$|\ln|\zeta - z| - h_0(z)| < (\mathrm{const})^n \min\{\delta^n/|z|^n;\ \delta/|z|\}, \quad |z| > \delta^2, \quad |\zeta| < \varepsilon,$$

and since

$$|z| = |z - \zeta + \zeta| > \tfrac{1}{2}|\zeta - z|,$$

if $|z| > \delta^2$, $|\zeta| < \varepsilon$, and if we take $\varepsilon < \delta^2/4$, then from the preceding we have

$$|\ln|\zeta - z| - h_0(z)| < (\mathrm{const})^n \min\{\delta^n/|\zeta - z|^n;\ \delta/|\zeta - z|\},$$

$$|\zeta - z| > 2\delta^2, \quad |\zeta| < \varepsilon.$$

Thus, for all ζ in the ε-neighborhood of the point 0 and, evidently, of any other point $\zeta_j \in R_\delta$, one may take the same harmonic polynomial $h_{\zeta_j}(z)$. The compact set E is covered by a finite number of disks $|\zeta - \zeta_j| < \varepsilon$, $j = 1, 2, \dots, N$. This ensures the possibility of choosing a finite number of distinct polynomials $h_\zeta(z)$, whose degrees, evidently, will be bounded in the aggregate. The lemma is proved.

Theorem 1. *If the compact set E has a connected complement, then every function $u(z)$, continuous on E, which at all interior points of E is infinitely differentiable and satisfies equation (1), can be expanded* into a uniformly convergent series on E of polynomials in z and $\bar{z}$ satisfying the very same equation.

Remark. The condition on $u(z)$ is necessary for the possibility of such an approximation. As for the condition on E, the author knows only that the interior of E must necessarily consist of simply connected components.

Proof. Let $k > 0,\ l > 0$. For an arbitrary number $\delta,\ 0 < \delta < 1$, by means of the operation of averaging the function $u(z)$, extended continuously to the whole plane, one constructs a function $u_\delta(z)$, continuously differentiable $(k+l)$ times (with respect to x and to y), possessing the properties:

1) $|u(z) - u_\delta(z)| \le \omega(\delta)$, where $\omega(\delta)$ is the modulus of continuity of the (extended) function $u(z)$ in the disk $|z| \le 2$;
2) $|D^k \bar{D}^l u_\delta(z)| < \mathrm{const} \cdot \omega(\delta)/\delta^{k+l},\ \ |z| \le 2$;
3) $D^k \bar{D}^l u_\delta(z) = 0,\ \ z \in E - R_\delta$.

We represent the function $u_\delta(z)$ in the form

$$u_\delta(z) = v(z) + \frac{1}{2\pi} \frac{(-1)^{k+l}}{2^{k+l-2}(k-1)!(l-1)!} \iint_G D^k \bar{D}^l u_\delta(\zeta)(\bar{\zeta} - \bar{z})^{k-1}(\zeta - z)^{l-1} \times$$

$$\times \ln|\zeta - z|\, d\xi\, d\eta, \tag{*}$$

where G is an arbitrary open set containing E and being the sum of a finite number of smooth simply connected domains, while the function $v(z),\ z \in G$, satisfies equation (1) in G, $\zeta = \xi + i\eta$.

By the already noted consequence of representation (2), the function $v(z)$ can be approximated on E uniformly with any accuracy by a polynomial solution of the same equation. As for the second term on the right-hand side of $(*)$, in view of property 3) and the arbitrariness of $G \supset E$, it suffices to be able to approximate the integral

$$I_\delta(z) = \iint_{R_\delta} D^k \bar{D}^l u_\delta(\zeta)(\bar{\zeta} - \bar{z})^{k-1}(\zeta - z)^{l-1} \ln|\zeta - z|\, d\xi\, d\eta.$$

For this purpose consider the integral

$$H_\delta(z) = \iint_{R_\delta} D^k \bar{D}^l u_\delta(\zeta)(\bar{\zeta} - \bar{z})^{k-1}(\zeta - z)^{l-1} h_\zeta(z)\, d\xi\, d\eta,$$

which, as follows from (2) and the last assertion of the lemma, is a polynomial solution of equation (1).

With the aid of the lemma (putting $n = k + l + 1$) and property 2) one can obtain the estimate

$$|I_\delta(z) - H_\delta(z)| \leq \mathrm{const} \cdot \omega(\delta), \qquad z \in E,$$

which, in view of property 1), completes the proof.

Next, two theorems are established on the rate of approximation by harmonic polynomials, analogous to the theorems of S. N. Mergelyan (^{3}) on best approximations by polynomials in z. In the first of them (see (^{4})) the question is that of approximation of a function harmonic and bounded inside the level line (L_R) of a continuum K having connected complement. By a level line is meant the preimage of the circle $|w| = R$ under a conformal mapping of the complement of K onto the disk $|w| > 1$, carrying ∞ to ∞. As before, we assume that K lies in the disk $|z| < 1$.

Theorem 2. If $f(z)$ is an analytic function of z inside the level line L_R of the continuum K and has there uniformly bounded real part

$$\sup |\operatorname{Re} f(z)| = M < \infty, \qquad z \text{ inside } L_R,$$

then for every natural number n there exists a polynomial $P_n(z)$ of degree n in z such that

$$|f(z) - P_n(z)| < \mathrm{const} \frac{M}{(R-1)^3} \frac{n^5}{R^{n-5}}, \qquad z \in K.$$

With the aid of the lemma and Theorem 2 one proves

Theorem 3. If a continuous function $g(z)$ is harmonic at all interior points of a continuum K with connected complement, then for every $n = 2, 3, 4, \ldots$ there exists a harmonic polynomial $H_n(z)$ of degree n such that

$$|g(z) - H_n(z)| < C\omega[d(\ln n/n)], \qquad z \in K,$$

where the constant C does not depend on n; $\omega(\delta)$ is the modulus of continuity of the function $g(z)$ in the disk $|z| \leq 2$, and the function $d(\lambda)$ is defined as in (3): $d(\lambda) = \max_{\zeta \in \partial K} \min_{z \in L_{1+\lambda}} |\zeta - z|$, and expresses the dependence of the maximum distance from points of the boundary ∂K of the set K to the level line $L_{1+\lambda}$.

The further formulation of the question is as follows: to describe the class of equations for which there is a positive answer to the question whether every solution of the equation can be approximated by functions satisfying the same equation, but in a wider domain.

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Rostov State University

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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