ON AN ANALOGUE OF A PROBLEM OF E. I. ZOLOTAREV

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Abstract

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ON AN ANALOGUE OF A PROBLEM OF E. I. ZOLOTAREV

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E. I. Zolotarev ([1]) posed and solved the following problem: among algebraic polynomials of the form $x^n - \sigma x^{n-1} + \cdots$, $0 \le \sigma < \infty$, find the one which deviates least from zero on the interval $[-1, 1]$. Its extremal polynomials turned out, for $0 \le \sigma \le n \operatorname{tg}^2 \frac{\pi}{2n}$, to be the polynomials

$$\frac{1}{2^{n-1}\alpha^n} T_n[(1+x)\alpha - 1], \quad T_n(x) = \cos n \arccos x, \quad \sigma = n\frac{1-\alpha}{\alpha}, \quad \alpha \in \left[\cos^2\frac{\pi}{2n}, 1\right],$$

and for $\sigma > n \operatorname{tg}^2 \frac{\pi}{2n}$, the Zolotarev polynomials $Z_n(\sigma, x)$.

The present note is devoted to the analogous problem for trigonometric polynomials $P_n(t)$

$$P_n(t) = \sum_{k=0}^{n} (a_k \cos kt + b_k \sin kt) = \sum_{k=-n}^{n} c_k e^{ikt},$$

where a_k and b_k are real; namely: among $P_n(t)$ whose coefficients a_n, b_n, a_{n-1}, and b_{n-1} are fixed, find the one which deviates least from zero on $[-\pi, \pi]$. To exclude elementary cases, we assume $n \ge 3$. It is clear that such a problem is equivalent to the problem: to construct those $P_n(t)$ which have on $[-\pi, \pi)$ at least $2n-2$ points of deviation and form at them a Chebyshev alternance. Since only $A\cos(nt - \psi)$, A, ψ real, have $2n$ points of deviation, the matter is the construction of the sets $K_{n,2n-1}$ and $K_{n,2n-2}$ of polynomials $P_n(t)$ which have on $[-\pi, \pi)$, respectively, exactly $2n-1$ or $2n-2$ points of deviation and form at them a Chebyshev alternance.

Theorem 1. *If $P_n(t)$ has on $[-\pi, \pi)$ exactly $2n-1$ points of deviation, then this is a polynomial of the form $AT_n(1 + \cos(t - \psi))\alpha - 1]$, $\alpha \in (\cos^2(\pi/2n), 1)$, A, ψ real.*

Suppose that on some interval (a, b) there are given $2n-1$ real functions $\{u_k(\xi)\}_0^n$ and $\{v_k(\xi)\}_1^n$, and let $\lambda_k(\xi) = u_k(\xi) + iv_k(\xi)$, $k = 0, 1, \dots, n$, $v_0(\xi) \equiv 0$, $\lambda_{-k}(\xi) \equiv \overline{\lambda_k(\xi)}$.

Taking any fixed $\xi \in (a,b)$, consider the set $K_n(L^\xi)$ of polynomials $P_n(t)$ whose coefficients are connected by the linear dependence L^ξ:

$$L^\xi(P_n) = \sum_{k=-n}^{n} c_k \lambda_k(\xi) = 1.$$

By $\pi_n(\xi,t)$ denote the polynomial least deviating from zero on $[-\pi,\pi]$ among the polynomials of the set $K_n(L^\xi)$. According to the criterion expressed by V. S. Videnskii ([2]), on the unit circle there exist points $\{\varepsilon_j(\xi)\}_1^m$, $\varepsilon_j(\xi) = e^{i\tau_j(\xi)}$, $-\pi \le \tau_1(\xi) < \tau_2(\xi) < \dots < \tau_m(\xi) < \pi$, $1 \le m \le 2n+1$, such that the system

$$\sum_{j=1}^{m} \varepsilon_j^k(\xi)\delta_j = \lambda_k(\xi), \quad k = 0, \pm 1, \dots, \pm n,$$

is consistent, and its solution $\{\delta_j(\xi)\}_1^m$ satisfies the conditions

$$\delta_j(\xi) \ne 0, \quad \operatorname{sign}\delta_j(\xi) = \operatorname{sign}\pi_n(\xi,\tau_j(\xi)), \quad j = 1, 2, \dots, m.$$

The proof of the following theorem is based on this property of the polynomial $\pi_n(\xi,t)$; in content and in the method of proof it is adjacent to the well-known theorem of S. N. Bernstein ([3]), p. 40).

Theorem 2. *If: 1) $\{u_k(\xi)\}_0^n$ and $\{v_k(\xi)\}_1^n$ are functions analytic on (a,b), and 2) the number m of points $\{\varepsilon_j(\xi)\}_1^m$ is constant on (a,b) and greater than n, then the coefficients $\{a_k(\xi)\}_0^n$ and $\{b_k(\xi)\}_1^n$ of the polynomial $\pi_n(\xi,t)$, as well as*

$$\pi_n(\xi) = \sup_{t\in[-\pi,\pi]} |\pi_n(\xi,t)|$$

are analytic functions on (a,b).

Let now $\lambda_k = 0,\ k = 0, 1, \dots, n-2,\ \lambda_{n-1} = -1,\ \lambda_n = w$, where w is a complex variable parameter. Taking any fixed w, consider the set $K_n(L^w)$ of polynomials $P_n(t)$ whose coefficients are connected by the linear relation

$$L^w:\ L^w(P_n) = \sum_{k=-n}^{n} c_k\lambda_k = 1, \quad \lambda_{-k} = \overline{\lambda_k}, \quad \text{i.e. } c_{nw} - c_{n-1} - c_{-n+1} + c_{-n}\overline{w} = 1.$$

Denote by $\pi_n(w,t)$ the polynomial least deviating from zero among the polynomials of the set $K_n(L^w)$; and by B_{2n-2} the set of points of the w-plane which is obtained if from the closed domain bounded by the rectilinear segments connecting the point $w = 0$ with the points

$$w = 2 - 2(n-1)\tan^2\frac{\pi}{2n}$$

and

$$w = 2\cos\frac{\pi}{2n} e^{i\pi/2(n-1)}$$

and by the curve

$$w = w(\psi) = e^{i(2n-1)\psi} + \left[1 - 2(n-1)\frac{\cos^2(n-1)\psi\cos^2\pi/2n}{\cos^2\pi/2n}\right]e^{i\psi}, \quad \psi \in \left[0, \frac{\pi}{2n(n-1)}\right] \tag{1}$$

the points of the curve (1) are excluded. The following simplest transformations of the trigonometric polynomial $P_n(t)$ are denoted by $(*)$: 1) multiplication of $P_n(t)$ by a real number; 2) replacement of the argument t by $t-\psi$, where ψ is real; 3) replacement of t by $2\pi - t$.

Theorem 3. *1) If $w \in B_{2n-2}$, then $\pi_n(w,t)$ is unique for every w and $\pi_n(w,t) \in K_{n,2n-2}$; 2) whatever polynomial $P_n^*(t) \in K_{n,2n-2}$ may be, there exists such a w^* in B_{2n-2} (moreover, only one for each $P_n^*(t)$) that $\pi_n(w^*,t)$ can be transformed into $P_n^*(t)$ using only $(*)$.*

By Theorems 2 and 3, the coefficients of the polynomial $\pi_n(w,t)$, as well as

$$\pi_n(w) = \sup_{t\in[-\pi,\pi]} |\pi_n(w,t)|,$$

are functions analytic in each of the arguments u and v, $w = u + iv$, for $w \in B_{2n-2}$. Therefore the deviation points of the polynomial $\pi_n(w,t)$, $w \in B_{2n-2}$, will be roots of $\partial\pi_n^*(w,t)/\partial w$ and $\partial\pi_n^*(w,t)/\partial v$, where $\pi_n^*(w,t) = \pi_n(w,t)/\pi_n(w)$.

Theorem 4. *The polynomial*

$$\pi_n^*(w,t) = \sum_{k=0}^{n}(a_k^*(u,v)\cos kt + b_k^*(u,v)\sin kt) = \sum_{k=-n}^{n} c_k^*(u,v)e^{ikt},$$

$$c_{-k}^*(u,v) = \overline{c_k^*(u,v)}, \quad w = u + iv, \quad w \in B_{2n-2},$$

satisfies in B_{2n-2} the equations

$$\frac{\partial\pi_n^*(w,t)}{\partial u} = \frac{1}{i}\,\frac{\dfrac{\partial c_n^*}{\partial u}e^{it} + \dfrac{\partial c_{n-1}^*}{\partial u} - w\dfrac{\partial c_n^*}{\partial u} - \dfrac{\partial c_{-n}^*}{\partial u}e^{-it}}{nc_n^*e^{it} + (n-1)c_{n-1}^* - nwc_n^* + nc_{-n}^*e^{-t}}\,\frac{\partial\pi_n^*(w,t)}{\partial t},$$

$$\frac{\partial\pi_n^*(w,t)}{\partial v} = \frac{1}{i}\,\frac{\dfrac{\partial c_n^*}{\partial v}e^{it} + \dfrac{\partial c_{n-1}^*}{\partial v} - w\dfrac{\partial c_n^*}{\partial v} - \dfrac{\partial c_{-n}^*}{\partial v}e^{-it}}{nc_n^*e^{it} + (n-1)c_{n-1}^* - nwc_n^* + nc_{-n}^*e^{-it}}\,\frac{\partial\pi_n^*(w,t)}{\partial t}. \tag{2}$$

Each of equations (2) splits into a system of first-order linear differential equations with respect to the coefficients $\{a_k^*(u,v)\}_0^n$ and $\{b_k^*(u,v)\}_1^n$ of the polynomial $\pi_n^*(w,t)$. One can indicate initial–

necessary conditions for integrating such a system. Thus, if w lies on the segment of the real axis between the points $w = 0$ and $w = 2 - 2(n-1)\operatorname{tg}^2 \frac{\pi}{2n}$, then $\pi_n^*(w, t)$ is known, namely:

$$\pi_n^*(w(\sigma), t) = \frac{1}{Z_n(\sigma)} Z_n(\sigma, \cos t),$$

where

$$Z_n(\sigma) = \sup_{x \in [-1,1]} |Z_n(\sigma, x)|, \qquad \sigma \in \left[n \operatorname{tg}^2 \frac{\pi}{2n}, \infty\right),$$

$$w(\sigma) = -2 \sum_{j=1}^{n} x_j(\sigma),$$

and $\{x_j(\sigma)\}_1^n$ are the deviation points of $Z_n(\sigma, x)$ on $[-1, 1]$.

When σ increases in $\left[n \operatorname{tg}^2 \frac{\pi}{2n}, \infty\right)$, $w(\sigma)$ moves along the indicated segment from

$$w\left(n \operatorname{tg}^2 \frac{\pi}{2n}\right) = 2 - 2(n-1) \operatorname{tg}^2 \frac{\pi}{2n}$$

to $w(\infty) = 0$. Moreover, at the points $w(\psi)$ of the curve (1),

$$\pi_n^*(w(\psi), t) = T_n\big[(1 + \cos(t - \psi))\alpha^*(\psi) - 1\big],$$

$$\psi \in \left[0, \frac{\pi}{2n(n-1)}\right], \qquad \alpha^*(\psi) = \frac{\cos^2 \pi/2n}{\cos^2(n-1)\psi}.$$

When ψ increases in $\left[0, \frac{\pi}{2n(n-1)}\right]$, $w(\psi)$ moves along the curve (1) from

$$w(0) = 2 - 2(n-1) \operatorname{tg}^2 \frac{\pi}{2n}$$

to

$$w\left(\frac{\pi}{2n(n-1)}\right) = 2 \cos \frac{\pi}{2n}\, e^{i\pi/2(n-1)}.$$

As follows from Theorem 3, every polynomial of the set $K_{n,2n-2}$ can be obtained by applying $(*)$ to some polynomial $\pi_n^*(w, t)$, $w \in B_{2n-2}$.

We note that, by a similar method, equations analogous to (2) for Zolotarev polynomials and certain other extremal algebraic polynomials were obtained by E. V. Voronovskaya ([4]).

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Note: Figure translations are in progress. See original paper for figures.