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ON A CLASS OF PROBLEMS OF MATHEMATICAL PHYSICS WITH A MIXED SPECTRUM OF EIGENVALUES

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Abstract

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MATHEMATICAL PHYSICS

I. T. Lozanovskaya, Ya. S. Uflyand

ON A CLASS OF PROBLEMS OF MATHEMATICAL PHYSICS WITH A MIXED SPECTRUM OF EIGENVALUES

(Presented by Academician B. P. Konstantinov, March 4, 1965)

In this work we investigate certain mixed problems for a semi-infinite interval in the one-dimensional case, when the sought function on a finite interval ($0 < x < l$) satisfies an equation of hyperbolic type, and on the remaining part of the interval—an equation of parabolic type. It is assumed here that at the point $x = l$ the function and its derivative may have finite discontinuities.*

1°. The simplest problem of the class under consideration consists in finding a function $u(x,t)$ satisfying the equations

$$\frac{\partial^2 u}{\partial x^2} = \alpha \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < l; \qquad \frac{\partial^2 u}{\partial x^2} = \beta \frac{\partial u}{\partial t}, \quad l < x < \infty, \tag{1}$$

and the conditions

$$u\big|_{x=0} = 0, \quad u\big|_{x=l-0} = \mu u\big|_{x=l+0}, \quad \partial u/\partial x\big|_{x=l-0} = \nu\, \partial u/\partial x\big|_{x=l+0},$$

$$u(\infty, t) < \infty, \tag{2}$$

$$u\big|_{t=0} = f(x), \quad 0 < x < \infty; \qquad \partial u/\partial t\big|_{t=0} = 0, \quad 0 < x < l.$$

Applying the Laplace transform

$$\bar{u}(x) = \int_0^\infty u(x,t) e^{-pt}\, dt, \tag{3}$$

we arrive at the equations

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$$\bar{u}'' - \alpha p^2 \bar{u} = -\alpha p f(x), \quad 0 < x < l; \qquad \bar{u}'' - \beta p \bar{u} = -\beta f(x), \quad l < x < \infty \tag{4}$$

and the boundary conditions

$$\bar{u}(0) = 0, \quad \bar{u}(l-0) = \mu \bar{u}(l+0), \quad \bar{u}'(l-0) = \nu \bar{u}'(l+0), \quad \bar{u}(\infty) < \infty. \tag{5}$$

The solution of this boundary-value problem can be expressed in the form**

$$\bar{u} = \frac{1}{D(p)} \int_0^\infty f(\xi) \Phi(x, \xi, p)\, d\xi; \tag{6}$$

$$\Phi = \begin{cases} \sqrt{\alpha}\,[\delta\sqrt{p}\ \mathrm{ch}\,\sqrt{\alpha}\,p(l-x) + \mathrm{sh}\,\sqrt{\alpha}\,p(l-x)]\ \mathrm{sh}\,\sqrt{\alpha}\,p\xi, & 0 < \xi < x < l, \\ \mu\sqrt{\dfrac{\beta}{p}}\ \mathrm{sh}\,\sqrt{\alpha}\,px\, e^{\sqrt{\beta p}(l-\xi)}, & 0 < x < l < \xi < \infty, \\ \dfrac{\alpha\sqrt{p}}{\nu\sqrt{\beta}}\ \mathrm{sh}\,\sqrt{\alpha}\,p\xi\, e^{\sqrt{\beta p}(l-x)}, & 0 < \xi < l < x < \infty, \\ \sqrt{\dfrac{\beta}{p}}\,[\mathrm{sh}\,\sqrt{\alpha}\,l\ \mathrm{ch}\,\sqrt{\beta p}(l-x) - \delta\ \mathrm{ch}\,\sqrt{\alpha}\,pl\ \mathrm{sh}\,\sqrt{\beta p}(l-x)]e^{\sqrt{\beta p}(l-\xi)}, & l < x < \xi < \infty, \end{cases} \tag{7}$$

* Problems of this kind arise, in particular, in the study of transient processes in piecewise-inhomogeneous media; see, for example, works ([1,5]), where the flow of electric fluid in a channel is considered, taking into account the conductivity of its walls, as well as note ([4]), devoted to the calculation of a composite electric line.

** For $x, \xi < l$ and $x, \xi > l$, $\Phi(\xi, x, p) = \Phi(x, \xi, p)$.

where

$$D(p) = \delta\sqrt{p}\ \mathrm{ch}\,\sqrt{\alpha}\,pl + \mathrm{sh}\,\sqrt{\alpha}\,pl, \qquad \mathrm{Re}\,\sqrt{p} > 0, \qquad \delta = \frac{\mu}{\nu}\sqrt{\frac{\alpha}{\beta}}. \tag{8}$$

By the inversion formula we obtain the solution of the posed problem in the form of the complex integral

$$u = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \bar{u} e^{pt}\, dp. \tag{9}$$

Since the singular points of the function $\bar{u}$ are the simple poles p_n—the roots of the equation $D(p) = 0$*, and also $p = 0$ (a branch point), the solution (9) is

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composed of the sum of residues and the integrals along the banks of the cut $p = -\lambda$ ($\lambda > 0$):

$$u(x,t) = 2\,\mathrm{Re}\sum_{n=1}^{\infty} \frac{e^{p_n t}}{D'(p_n)} \int_0^{\infty} f(\xi)\Phi(x,\xi,p_n)\,d\xi -$$

$$-\frac{1}{\pi}\,\mathrm{Im}\int_0^{\infty} \frac{e^{-\lambda t}}{D(\lambda e^{i\pi})}\,d\lambda \int_0^{\infty} f(\xi)\Phi(x,\xi,\lambda e^{i\pi})\,d\xi \qquad (\mathrm{Im}\,p_n > 0). \qquad (10)$$

2°. To the mixed problem of mathematical physics considered there corresponds the spectral problem**

$$y'' + \psi(x,\lambda)y = 0; \qquad \psi(x,\lambda) = \begin{cases} -\alpha\lambda^2, & 0 < x < l, \\ \beta\lambda, & l < x < \infty; \end{cases} \qquad (11)$$

$$y(0) = 0, \qquad y(l-0) = \mu y(l+0), \qquad y'(l-0) = \nu y'(l+0), \qquad y(\infty) < \infty,$$

which has a mixed spectrum of eigenvalues, consisting of points of the real axis $\lambda > 0$ and the roots $\lambda_n = -p_n$ of the equation

$$\omega(\lambda) = \delta\sqrt{-\lambda}\,\mathrm{ch}\,\sqrt{\alpha}\lambda l - \mathrm{sh}\,\sqrt{\alpha}\lambda l, \qquad \mathrm{Re}\,\sqrt{-\lambda} > 0. \qquad (12)$$

With the aid of the solution found in (10), one can indicate a formula for expanding a given function $f(x)$ in the eigenfunctions of the boundary-value problem (11). If the eigenfunctions are defined by the formulas

$$y_n(x) \begin{cases} \mu\,\dfrac{\mathrm{sh}\,\sqrt{\alpha}\lambda_n x}{\mathrm{sh}\,\sqrt{\alpha}\lambda_n l}, & 0 < x < l, \\ e^{\sqrt{-\beta\lambda_n}(l-x)}, & l < x < \infty; \end{cases} \qquad (13)$$

$$y(x,\lambda) = \begin{cases} \mu\,\mathrm{sh}\,\sqrt{\alpha}\lambda x, & 0 < x < l, \\ \mathrm{sh}\,\sqrt{\alpha}\lambda l \cos\sqrt{\beta\lambda}(l-x) - \delta\sqrt{\lambda}\,\mathrm{ch}\,\sqrt{\alpha}\lambda l \sin\sqrt{\beta\lambda}(l-x), & l < x < \infty, \end{cases} \qquad (14)$$

and carry out in (10) the limiting passage as $t \to 0$, then, after some transformations, we obtain the desired expansion formula***

$$f(x) = 2\,\mathrm{Re}\sum_{n=1}^{\infty} \frac{\mathrm{sh}\,\sqrt{\alpha}\lambda_n l}{\omega'(\lambda_n)} y_n(x) \int_0^{\infty} f(\xi) y_n(\xi) r_n(\xi)\,d\xi +$$

* It can be proved that this equation has an infinite set of complex-conjugate simple roots, for which $\operatorname{Re}\sqrt{p} >$

** Problems of this type apparently have not been considered in the literature.

*** Establishing the class of functions for which formula (15) is valid will be the subject of a separate investig

$$+\frac{1}{\pi}\int_0^\infty \frac{y(x,\lambda)}{|\omega(\lambda)|^2}\,d\lambda \int_0^\infty f(\xi)y(\xi,\lambda)r(\xi,\lambda)\,d\xi \qquad (\operatorname{Im}\lambda_n > 0), \tag{15}$$

where

$$r_n(x) = \frac{\delta}{\mu^2}\sqrt{-\alpha\lambda_n}, \quad 0 < x < l; \qquad r_n(x) = \sqrt{-\frac{\beta}{\lambda_n}}, \quad l < x < \infty; \tag{16}$$

$$r(x,\lambda) = -\frac{\delta}{\mu^2}\sqrt{\alpha\lambda}, \quad 0 < x < l; \qquad r(x,\lambda) = \sqrt{\frac{\beta}{\lambda}}, \quad l < x < \infty. \tag{17}$$

3°. In an analogous way, the corresponding expansions may be found for boundary conditions of the second or third kind at $x = 0$, as well as for more complicated conditions at the point $x = l$ (for example, those containing derivatives with respect to time).

By the method set forth, one can also establish the mixed character of the spectrum for boundary-value problems connected with the equation

$$\frac{\partial^2 u}{\partial x^2} = \alpha\frac{\partial^2 u}{\partial t^2} + \beta\frac{\partial u}{\partial t} + \gamma u, \qquad 0 < x < \infty,$$

where α, β, γ are piecewise constant coefficients. Exceptions may be some limiting cases, for example $\beta = \gamma \equiv 0$, when the spectrum is discrete, or $\alpha = \gamma \equiv 0$ (continuous spectrum). A continuous spectrum is also obtained in the case when, in (1), the equation of hyperbolic type takes place for $-\infty < x < 0$ (see (2)).

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Note: Figure translations are in progress. See original paper for figures.

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