V. A. Rokhlin

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Abstract

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V. A. Rokhlin

NEW EXAMPLES OF FOUR-DIMENSIONAL MANIFOLDS

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1. Introduction.

In this note a *manifold* means an oriented smooth connected compact manifold, and the word *submanifold* has the analogous meaning. We study simply connected closed four-dimensional manifolds. If M is such a manifold, then the self-intersection index of an element of the integral group $H_2(M)$ is a quadratic form on $H_2(M)$ with discriminant ± 1, denoted by $q(M)$. Considered up to isomorphism, the form $q(M)$ is an invariant of the oriented homotopy type of the manifold M and, as was proved in 1949 by Whitehead ([1]) and L. S. Pontryagin ([2]), completely determines this type. Subsequently ([3–5]), the invariants of the tangent bundle of the manifold M were also expressed in terms of $q(M)$, and recently S. P. Novikov ([6]) and Wall ([7]) proved that $q(M)$ determines M up to h-cobordism. Obviously,

$$q(M_1 \# M_2) \simeq q(M_1) \oplus q(M_2).$$

The following facts from algebra make it possible to estimate the degree of effectiveness of the preceding classification theorems. We shall agree to call integral quadratic forms with discriminant ± 1 simply *forms*, and to assign a form to type II if all its values are even, and to type I otherwise. It is known that there exists only a finite number of nonisomorphic forms of a given rank; that an indefinite form is determined up to isomorphism by its rank, signature, and type; and that the signature of a form of type II is divisible by 8 (see, for example, ([8],[9])). Further, a positive form decomposes in a unique way into an orthogonal sum of indecomposable forms ([10],[11]), and there exists an infinite number of indecomposable positive forms both of type I and of type II—for example, there is a construction assigning to each $r \equiv 0 \pmod 4$ and > 4 an indecomposable positive form K_r, belonging to type II if and only if $r \equiv 0 \pmod 8$ ([12]). There is no satisfactory classification of positive forms; only a list of indecomposable positive forms of ranks ≤ 16 has been compiled: there exist, up to isomorphism, 7 such forms, their ranks are $1, 8, 12, 14, 15, 16, 16$, and only two of them, K_8 and K_{16}, belong to type II ([13]).

The main unsolved problem in the homotopy theory of simply connected four-dimensional manifolds is which forms are realizable by manifolds, i.e., what must a form q be like for there to exist a (simply connected closed) manifold M with $q(M) \simeq q$. Until now this problem had apparently been considered only in two papers ([4],[8]). In my note ([4]) it was proved that the signature of a realizable form of type II is divisible by 16 and that realizable forms of type II with signature 16 exist, while in Milnor' s survey ([8]) manifolds with a form of type II, signature 16 and rank 22 are indicated, and the existence of a manifold with a *positive* form of type II and signature 16 is put forward as a problem of first priority. (I note that the form of type II and signature 16, implicitly realized in ([4]), also had rank 22; the construction of the realizing manifold is reproduced below in § 6.) For completeness I shall also list the trivial facts: connected sums of manifolds diffeomorphic to P_2C, $\overline{P_2C}$, and $S^2 \times S^2$ realize all forms

$$I_p \oplus (-I_n),$$

where I_p is the orthogonal sum of p positive forms

rank 1 (in particular, all indefinite forms of type I) and all forms of type II with signature 0.

2. Formulation of results. *The forms $K_8 \oplus I_1$ and $K_8 \oplus K_8$ are realizable. Every indefinite form of type II with signature divisible by 16 is realizable.*

The last assertion exhausts the problem of realizability of indefinite forms. It is a consequence of the realizability of the form $K_8 \oplus K_8$: if B is a manifold realizing this form, then an indefinite form of type II with signature $16s \geqslant 0$ and rank r is realized by the manifold $sB\#\frac{1}{2}(r-16s)(S^2 \times S^2)$, while the case $s < 0$ reduces to the case $s > 0$.

The forms $K_8 \oplus I_1$ and $K_8 \oplus K_8$ will be realized below. Since they do not decompose into realizable forms, the manifolds A and B realizing them do not decompose into connected sums not containing homotopy spheres; thus, to the three known manifolds indecomposable in this sense, P_2C, $\overline{P_2C}$, and $S^2 \times S^2$, new ones are added. I also note that in the class of realizable positive forms there is no uniqueness of decomposition into indecomposable ones: for example, $(K_8 \oplus I_1) \oplus (K_8 \oplus I_1) = (K_8 \oplus K_8) \oplus I_1 \oplus I_1$. It is unknown whether the manifolds $A\#A$ and $B\#P_2C\#P_2C$ are diffeomorphic.

3. Membranes. Let M be a four-dimensional manifold and let F be its closed two-dimensional submanifold. A **membrane** on F is any submanifold P of the manifold M, diffeomorphic to a disk, intersecting F exactly in its boundary ∂P and having with F (along ∂P) no common tangent planes. If one constructs on ∂P a vector field tangent to F and not tangent to P, then the attempt to extend it without touching onto P leads to a two-dimensional obstruction with integer index (the orientations used of planes transversal to P must be compatible with the orientations of P and M), independent of the choice of the field on ∂P. This index is called the **index of the membrane** P and is denoted by $i(P)$. It does not depend on the orientation of P and changes sign when the orientation of M is changed.

If $i(P) = 0$, then the field extends onto P without touching. In this case one can perform a Morse surgery of the submanifold F along P: replace the cylinder serving as a neighborhood of the circle ∂P in F by two nonintersecting smoothly attached disks close to P. If ∂P does not separate F, then the result will be a decrease of the genus of the surface F by one.

The following two methods make it possible to rebuild membranes. Let Σ be a submanifold of the manifold M, diffeomorphic to S^2, and not intersecting P. The first method assumes that Σ does not intersect F, and consists in forming the connected sum of P and Σ (along some path). The result is a new membrane P_1 with $\partial P_1 = \partial P$ and $i(P_1) = i(P) + \xi^2$, where ξ is the element of the group $H_2(M)$ represented by the sphere Σ. The second method assumes that Σ intersects F at one point, and moreover regularly and with index $+1$, and is carried out in two steps. First step: the sphere Σ is deformed in a neighborhood of its intersection with F so that the intersection becomes a circle, and the closed complement of this circle in Σ becomes a membrane. Second step: this membrane is connected with P (along some path lying on F) into a new membrane P_1. The first step can be carried out in two homologically different ways: in one way the degree v of the normal mapping of the sphere Σ onto F is $+1$, in the other -1. As is not difficult to show, $i(P_1) = i(P) + \xi^2 + v$, and it is clear that the circles ∂P and ∂P_1 are homologous on F.

4. Example. Put $M = P_2C$ and take for F the torus defined in homogeneous coordinates x, y, z by the equation $y^2 z = x^3 - z^3$ and representing the tripled generator of the group $H_2(P_2C)$. The purpose of this paragraph is to construct on F two membranes P_1, P_2 of index 1, lying in the cell $z \neq 0$ and intersecting in one point, which serves as a regular point of intersection of their boundaries on F.

Since we restrict ourselves to the Euclidean part $z \neq 0$ of the manifold P_2C, we may assume that F is defined by the equation $y^2 = x^3 - 1$ in complex Cartesian coordinates x, y. Let K be a circle in the plane $y = 0$, containing two roots of the equation $x^3 = 1$ as interior points and containing the third root neither in its interior nor on its boundary, and let L be one of the two loops on F covering the circle ∂K. Construct on L, as on a base, a cone with vertex $x = y = 0$, and smooth it near the vertex (which is not a point of local knotting). Then the cone turns into a membrane P_1 of index 1, and, rotating it by 120° about the plane $x = 0$, we obtain a membrane P_2. The boundaries of these membranes intersect at one point, regularly on F, and the intersections not lying on the boundaries are removed by a small deformation.

5. **Realization of the form $K_8 \oplus I_1$.** Put $X = \overline{P_2C} \# 9 P_2C$, and let $\Sigma_0, \Sigma_1, \dots, \Sigma_9$ be projective lines in the summands and $\alpha_0, \alpha_1, \dots, \alpha_9$ the generators of the group $H_2(X)$ determined by them. Put, further, $\beta = 3\alpha_0 + \alpha_1 + \dots + \alpha_8$, $\gamma = \beta + \alpha_9$, and denote by G the annihilator of the class β in $H_2(X)$ (with respect to the intersection index), and by G_1 the annihilator of the pair β, α_9. Since $\beta^2 = -1$, $\alpha_9^2 = 1$, $\beta\alpha_9 = 0$, and the form $q(X)$ has rank 10 and signature 8, it follows that $H_2(X) = G \oplus Z\beta =$

$G_1 \oplus Z\alpha_9 \oplus Z\beta$, and on G and G_1 the form $q(X)$ is positive and has discriminant 1 and ranks 9 and 8. Since, moreover, $\xi^2 \equiv \xi\gamma \mod 2$ for every $\xi \in H_2(X)$, the form $q(X)|_{G_1}$ belongs to type II. Consequently, $q(X)|_{G_1} \simeq K_8$, and $q(X)|_G \simeq K_8 \oplus I_1$.

The connected sum E of the torus $y^2z = x^3 - z^3$, representing the class $3\alpha_0$ in $\overline{P_2C}$ (see § 4), and the spheres $\Sigma_1, \dots, \Sigma_8$ is a torus representing the class β in X. The membrane P_1, constructed in § 4, may be regarded as a membrane on E; it has index -1, and its boundary does not separate E. Reconstructing it with the aid of the sphere Σ_9 (the first method of § 3), we obtain a membrane of index $-1+\alpha_9^2 = 0$ with the same boundary. Consequently, the class β is represented by a sphere. Since $\beta^2 = -1$, the boundary of a tubular neighborhood of this sphere is diffeomorphic to S^3. Replacing this neighborhood by a four-dimensional ball, we obtain a simply connected manifold A with $q(A) \simeq q(X)|_G \simeq K_8 \oplus I_1$.

Remark. The manifold $A\#\overline{P_2C}$ is diffeomorphic to X.

6. **Realization of the form $K_8 \oplus K_8$.** Return to the manifold X. The connected sum T of the torus E and the sphere Σ_9 is a torus representing the class γ. Since $\gamma^2 = 0$, the boundary of a tubular neighborhood of the torus T has the form $T \times S^1$. Cutting it out, we obtain a simply connected manifold W with boundary $\partial W = T \times S^1$ and with a form isomorphic to K_8 (the form of a four-dimensional manifold M with boundary is defined as the self-intersection index, considered on $H_2(M)/i_*H_2(\partial M)$, where $i : \partial M \to M$ is the inclusion; if M is simply connected, then i_* is a monomorphism). The membranes P_1, P_2, constructed in § 4, may be regarded as membranes on T in X, and in W from them there remain disks D_1, D_2, while from the sphere Σ_9 there remains a disk D_3. The generators of the group $H_2(\partial W)$ are represented by the tori $T_1 = \partial P_2 \times S^1$, $T_2 = \partial P_1 \times S^1$, $T_3 = T \times c$, where $c \in S^1$. The intersections $D_1 \cap T_1$, $D_2 \cap T_2$, $D_3 \cap T_3$ are one-point and regular, and the disks D_1, D_2, D_3 become, after a suitable displacement, membranes: D_1 a membrane of index -1 on T_2 and T_3; D_2 a membrane of index -1 on T_3 and T_1; D_3 a membrane of index 1 on T_1 and T_2. The boundaries of these membranes do not separate T_1, T_2, T_3.

Glue the manifold W to its second copy by means of the automorphism of the boundary $\partial W = T \times S^1$ composed of the identity transformation of the torus T and the reflection of the circle S^1 in its diameter. We obtain a closed simply connected manifold Y with a form $q(Y)$ of type II, of rank 22 and signature 16, in which the disks D_1, D_2, D_3 and their doubles D_1', D_2', D_3' are smoothly glued into spheres S_1, S_2, S_3. The nonzero intersection indices of the elements $\sigma_1, \sigma_2, \sigma_3, \tau_1, \tau_2, \tau_3$ of the group $H_2(Y)$, represented by the spheres S_1, S_2, S_3 and the tori T_1, T_2, T_3, are given by the table $\sigma_1^2 = \sigma_2^2 = -2$, $\sigma_3^2 = 2$, $\sigma_1\tau_1 = \sigma_2\tau_2 = \sigma_3\tau_3 = 1$, and on the annihilator G_2 of this sextuple of classes the form $q(Y)$ is isomorphic to $K_8 \oplus K_8$.

We reconstruct on the torus T_1 the membrane D_3 by means of the sphere S_1, on the torus T_2 the membrane D_3 by means of the sphere S_2, and on the torus T_3 the membrane D'_1 by means of the sphere S_3, putting $\nu = 1, 1, -1$ (the second method of § 3). We obtain membranes with indices

$$i(D_3) + \sigma_1^2 + 1 = 0, \qquad i(D_3) + \sigma_2^2 + 1 = 0,$$

$$i(D'_1) + \sigma_3^2 - 1 = 0$$

and boundaries that do not split T_1, T_2, T_3. Consequently, the classes τ_1, τ_2, τ_3 are represented by spheres. These spheres are easily made disjoint by moving apart, before the reconstructions, the tori T_1, T_2, T_3, the spheres S_1, S_2, S_3, and the membranes; and since $\tau_1^2 = \tau_2^2 = \tau_3^2 = 0$, one can perform Morse reconstructions along them, replacing their nonintersecting tubular neighborhoods by products of a three-dimensional ball with a circle. As a result, Y is transformed into a simply connected manifold B with form

$$q(B) \simeq q(Y)|_{G_2} \simeq K_8 \oplus K_8.$$

Remark. The manifold $B\#3(S^2 \times S^2)$ is diffeomorphic to Y.

Leningrad State University
named after A. A. Zhdanov

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REFERENCES

1. J. H. C. Whitehead, Comm. Math. Helv., 22, 48 (1949).
2. L. S. Pontryagin, Uspekhi Mat. Nauk, 4, no. 4, 157 (1949).
3. Wu Wen-tsun, C. R., 230, 508 (1950).
4. V. A. Rokhlin, DAN SSSR, 84, 221 (1952).
5. R. Thom, Comm. Math. Helv., 28, 17 (1954).
6. S. P. Novikov, Izv. AN SSSR, Ser. Math., 28, no. 2 (1964).
7. C. T. C. Wall, J. London Math. Soc., 39, 141 (1964).
8. J. Milnor, Symposium International de Topologia Algebraica, Mexico, 1958, p. 122.

9. J.-P. Serre, Séminaire Henri Cartan, 1961/62, no. 14–15.

10. M. Eichler, Math. Ann., 125, 51 (1952).

11. M. Kneser, Math. Ann., 127, 105 (1954).

12. O. T. O' Meara, Introduction to Quadratic Forms, 1963.

13. M. Kneser, Arch. Math., 8, 241 (1957).

Note: Figure translations are in progress. See original paper for figures.