Mathematics

1965

SovietRxiv

Abstract

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Mathematics

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Conformal-Metric Theory of Doubly Connected Domains and the Generalized Blaschke Product*

(Presented by Academician M. A. Lavrent' ev, 10 X 1964)

Several years ago it became clear that the solution of a number of problems in the theory of functions of a complex variable requires a more complete knowledge of the conformal-metric properties of multiply connected domains, whose theory is poor in concrete facts and estimates. In this connection V. A. Zmorovich put forward a hypothesis on the boundedness of one functional characterizing the distortion of the boundary of a doubly connected domain under a univalent conformal mapping.

In papers ([1]-[4]) the author obtained a number of qualitative and quantitative results on the conformal mapping of doubly connected domains; in particular, V. A. Zmorovich' s hypothesis was proved. Relying on papers ([1]-[4]), one can establish a new proposition.

Denote by $\mathfrak{H}^*(R)$ the class of univalent conformal mappings $f(z)$ of the ring $K_R : R < |z| < 1$ into the disk $|z| < 1$, under which the unit circle is mapped onto itself. For $f \in \mathfrak{H}^*(R)$ let $P(f, R)$ and $Q(f, R)$ denote, respectively, the maximum and the minimum of the quantity $|1-w|$, when w runs over the inner boundary of the domain $f(K_R)$.

Theorem 1. *There exist finite constants $m(R)$ and $N(R)$, depending only on the number R, such that for any two mappings from $\mathfrak{H}^*(R)$ $(0 \le R < 1)$, connected by the normalization $f_1 f_2^{-1}(1) = 1$, under the condition $f_1 f_2^{-1}(0) = 0$ the inequality*

$$\frac{P(f_1, R)}{Q(f_2, R)} < m(R), \tag{1}$$

holds, and under the condition $w = 0 \in f_1(K_R) \cup f_2(K_R)$ the inequality

$$\frac{P(f_1, R)}{Q(f_2, R)} < N(R) \tag{2}$$

holds.

The quantity $m(R)$ is given by the formula:

$$m(R) = \max\left\{ N(R)\frac{1+\sin\beta(R)}{1-\sin\beta(R)};\quad N(R)\frac{1+\sin\beta(R)}{-\cos\beta(R)} + N_1(R)\cos\beta(R) + \right.$$
$$\left. + \max\left\{ \frac{\exp\left[(\sqrt{5}+1)\sqrt{j(R)\left(j(R)+\frac{\sqrt{5}-1}{2}\right)}\right]}{\sin\left[\frac{1}{2}\beta(R)-\pi/4\right]};\ \frac{2}{1-\sin\beta(R)} \right\} \right\}, \tag{3}$$

where $N(R)$, $N_1(R)$, $\beta(R)$, and $j(R)$ are defined by relations (13) and (19) of paper ([1]), (3) and (19) of paper ([3]), and relation (6) of paper ([4]).

* Reported at a meeting of the Scientific Council of the Institute of Mathematics of the Academy of Sciences of the Ukrainian SSR, 12 II 1963.

The proposition formulated is an essential generalization of the core of Theorem 2 of paper (2), which confirmed the validity of V. A. Zmorovich' s hypothesis.

The estimates of Theorem 1 and of papers (1-4) pertain to classes of mappings, not to individual mappings; moreover, the corresponding bounds in these estimates are finite and depend only on the modulus of the doubly connected domain. In the literature one usually studies problems that are essentially connected with classes of conformal mappings of a doubly connected domain that are compact in themselves. In this case the existence of extremal mappings and the boundedness of various functionals are automatic consequences of the compactness principle. In contrast, our main results pertain to classes of mappings of a variable doubly connected domain that are not compact in themselves. Here the principal question is the boundedness of one or another functional on the indicated classes. Such a point of view is equivalent to considering functionals defined on pairs of mappings from classes of mappings of a constant doubly connected domain that are not compact in themselves.

From the results obtained up to the present time one may conclude that the various conformal deformations of a doubly connected domain are confined within strict limits determined solely by the modulus of the domain. In this sense one may speak of the relative rigidity of a doubly connected domain under a univalent conformal mapping. This rigidity plays an important role in certain applications of the theory of conformal mappings, in particular in the theory of the generalized Blaschke function in multiply connected domains.

The range of applicability of the estimates obtained is not limited to doubly connected domains. It is not difficult to see that these estimates carry over to domains of arbitrary connectivity satisfying the single additional condition that the marked boundary component be isolated (for example, all finitely connected domains obviously satisfy the indicated condition). Thus, in analogous problems for multiply connected domains with an isolated marked boundary component there is nothing new in comparison with the doubly connected case.

Let us also note the following circumstance. G. Grötzsch showed that theorems on univalent conformal mapping onto canonical domains extend in a natural way to quasiconformal mappings (5, 6). The transfer of known results on conformal mapping to quasiconformal mappings (for infinitely connected domains) is also the subject of works by I. P. Mityuk (7, 8). In this respect, our results too can be generalized to quasiconformal mappings.

In paper (9) the question of convergence of the generalized Blaschke product was studied under very broad conditions on the structure of the initial domain, the form of the univalent factors, and the type of their angular normalization. It was very useful to abandon a rigid angular normalization and to introduce a flexible angular normalization (of two different types). Subsequently it turned out that the normalization introduced essentially cannot be extended further. At the same time it was unexpectedly found that the sufficient conditions for convergence of the generalized Blaschke product that had been considered, thanks to the flexibility of the angular normalization, are also necessary. We formulate the final result. For simplicity of formulation we restrict ourselves here to the case where on the boundary of the domain R only one boundary component is marked, which in addition is the outer boundary of the domain B and coincides with the unit circle C.

Let $a \in B$, and let $\mathfrak{B}(B,a)$ be the class of univalent conformal mappings $f(z)$ of the domain B into the unit disk, under which C passes to the outer boundary of the image and $f(a)=0$. Let $\mathfrak{B}^*(B,a)$ be the class of mappings ···

mappings $f \in \mathfrak{B}(B,a)$ that map C into itself. For $f \in \mathfrak{B}(B,a)$, by $s(f,B)$ and $S(f,B)$ we shall denote respectively $\inf|\arg w|$ and $\sup|\arg w|$ on the inner boundary of the domain $f(B)$ (i.e., on that part of the boundary of the domain $f(B)$ which lies inside the unit disk). We shall also denote

$$\varkappa(B,a) = \inf_{f\in\mathfrak{B}^*(B,a)} |f'(a)|; \qquad l(f,B) = \frac{|f'(a)|}{\varkappa(B,a)}. \tag{4}$$

If D is a doubly connected domain with outer boundary C, if $a \in D$ and $f \in \mathfrak{B}^*(D,a)$, then by $\sigma(f,D)$ we shall denote the principal value of the argument of the epicenter $E[0,C,f(D)]$ (for the definition of the epicenter see (9)).

Theorem 2. *Let B be a domain of arbitrary connectivity whose outer, and moreover isolated, boundary component is the unit circle C; let D be an arbitrary fixed doubly connected subdomain (of the domain B) with outer boundary C; let $\{a_n\}$ be a sequence of points of the domain B; let $\{f_n(z)\}$ be a sequence of mappings $f_n \in \mathfrak{B}(B,a_n)$ satisfying the condition*

$$l(f_n,B) \geq 1, \qquad n=1,2,\dots. \tag{5}$$

Then, among the following five assertions, any two (in which all occurring series have meaning) are equivalent:

1) *The product*

$$\prod_{n=1}^{\infty} f_n(z_0)$$

converges absolutely in the generalized sense (see (10), p. 210) at some fixed point $z_0 \in B$.

2) *The series*

$$\sum_{n=1}^{\infty}(1-|a_n|) \quad \text{and} \quad \sum_{n=1}^{\infty} s(f_n, B)$$

converge.

3) *The series*

$$\sum_{n=1}^{\infty}(1-|a_n|) \quad \text{and} \quad \sum_{n=1}^{\infty} S(f_n, B)$$

converge.

4) *There exists a natural number* n_1 *such that the series*

$$\sum_{n=1}^{\infty}(1-|a_n|)$$

and

$$\sum_{n=n_1}^{\infty} |\sigma(f_n, D)|$$

converge.

5) *The product*

$$\prod_{n=1}^{\infty} f_n(z)$$

converges in the generalized sense to an analytic function $f(z)$, *absolutely in* B *and uniformly on every part of the domain* B *whose distance from* C *is positive. Moreover,* $|f(z)| < 1$ *in* B, *and* $f(z)$ *has there no zeros other than* a_n

$(n = 1, 2, ...)$. *Furthermore, on every part of the domain B whose distance from the set*

$$C \cup \left(\bigcup_{n=1}^{\infty} a_n \right)$$

is positive, $f(z)$ is bounded below in modulus by a positive number. In particular, the values of $f(z)$ on the inner boundary of the domain B are bounded below in modulus by a positive number.

The series entering into assertions 2) and 3) have meaning if and only if the domain B is finit connected. The series entering into assertion 4) have meaning if and only if $a_n \in D$ and $f_n \in \mathfrak{B}^*(B, a_n)$ for all $n \geq n_1$.

The proof that each of assertions 1) and 5) implies the validity of assertions 2), 3), and 4), and also the proof that each of assertions 2), 3), and 4) implies the validity of assertions 1) and 5), are based on the conformal-metric properties of doubly connected domains studied by the author.

Theorem 2 remains valid if, in its formulation, condition (5) is replaced by the following more general condition:

$$\sum_{n=1}^{\infty} \sqrt{[1 - l(f_n, B_0)]^+} < +\infty, \tag{6}$$

where B_0 is an arbitrary fixed subdomain of the domain B, for which the circle C is an isolated boundary component. On the other hand, if the series on the left-hand side of (6) diverges, then Theorem 2 loses its force; moreover, under the condition

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{\sqrt{l(f_n, B_0)}} = +\infty$$

the product

$$\prod_{n=1}^{\infty} f_n(z)$$

always diverges.

The basic properties of the generalized Blaschke product are described in the paper ([9]).

Received
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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.