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Abstract

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ON A PROBLEM OF STOCHASTIC LINEAR PROGRAMMING AND ITS STABILITY

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A problem of stochastic linear programming is considered in the following formulation.

It is required to minimize the linear function $F(\mathbf{x})$, $\mathbf{x} = x_1, \dots, x_n$, under the linear constraints $\sum_j a_{ij}x_j \leqslant b_i(\xi)$, $i = 1, \dots, m$, where $b_i(\xi)$ are independent random variables with mathematical expectation b_i and variance σ_i^2.

The constraints $\sum_j a_{ij}x_j \leqslant b_i$ will be called **constraints on the average**, and the corresponding linear programming problem—the problem on the average. For each realization of the random variable $b_i(\xi)$, as the i-th constraint there appears a hyperplane $\sum_j a_{ij}x_j = b_i(\xi)$ of dimension $n-1$, parallel to the hyperplane on the average with the same index.

The intersection of the half-spaces $\sum_j a_{ij}x_j \leqslant b_i(\xi)$ forms a random region D. By Chebyshev' s inequality,

$$\mathbf{P}\left\{|b_i(\xi) - b_i| < l\sigma_i\right\} > 1 - \frac{1}{l^2}.$$

The probability that the inequality $|b_i(\xi) - b_i| < l\sigma_i$ holds for all $i = 1, \dots, m$ is greater than $(1 - 1/l^2)^m$, by virtue of the independence of the $b_i(\xi)$. Choose l from the condition $(1 - 1/l^2)^m > 1 - \varepsilon$, where ε is the prescribed significance level.

The constraints $\sum_j a_{ij}x_j \leqslant b_i - l\sigma_i$ will be called **lower**, and $\sum_j a_{ij}x_j \leqslant b_i + l\sigma_i$ **upper**. The region of fulfillment of the lower constraints $i = 1, \dots, m$ will be denoted by $\underline{D}$, and of the upper constraints by $\overline{D}$. Let $\underline{D}$ be nonempty. Obviously, for any realization of the constraints, except for a set of measure ε, $\underline{D} \subset D \subset \overline{D}$. Thus, for each realization modulo ε there exists at least one feasible plan.

We note that under a parallel displacement of the constraints, their intersections of any dimension are displaced parallel to themselves. If not a single one-dimensional intersection of the constraints on the average is parallel to the hyperplanes of the level $F(\mathbf{x}) = C$ of the objective function, then the solution of the linear problem is unique for any realization of the constraints. We shall assume this condition to be satisfied. Each vertex of the polyhedron D will be denoted by the set $i_1, \dots, i_n$ of indices of those faces by whose intersection it is formed. The vertex on the average with the same indices will be denoted by $\overline{i_1, \dots, i_n}$. The true vertex $i_1, \dots, i_n$ is a random point; its mathematical expectation is the point $\overline{i_1, \dots, i_n}$, since finding the intersection of hyperplanes is a linear operation.

The vertex $i_1, \dots, i_n$, with probability greater than $(1 - 1/q^2)^n$, lies inside the ellipsoid whose center is the point $\overline{i_1, \dots, i_n}$, and along the normal to the i_k-th constraint ($k = 1, \dots, n$) the quantity $q\sigma_{i_k}$ is laid off. The number q is determined by the inequality $(1-1/q^2)^n > 1-\varepsilon$. We shall call the ellipsoid thus constructed the **ellipsoid of dispersion** of the point $i_1, \dots, i_n$. Sometimes the distribution of the vertex can be written in closed form. For example, if all

If the $b_i(\xi)$ are normally distributed, then the vertex whose coordinates are the solution of the linear equation $Ax = B(\xi)$ is normally distributed: $x = A^{-1}B(\xi)$ is a linear transformation of normally distributed quantities. The multidimensional density is easily written in terms of the numbers b_i, σ_i, and the gradients of the constraints.

Solving the linear programming problem on average gives an extremum on average. The hyperplanes $\sum_j a_{ij}x_j = b_i$, on which the point of the extremum on average lies, will be called marked. In the linear problem there are n marked constraints. In the quadratic problem there may be from 1 to n of them, depending on the rank of the augmented matrix of the quadratic form. A detailed formulation of the problem of quadratic stochastic programming is not included in this article. However, it is expedient to give the definition of stochastic stability of the solution of a convex programming problem in a form convenient simultaneously for linear and quadratic programming.

We shall call a solution of a convex programming problem **stochastically stable modulo** ε if the set of indices of the marked constraints is constant for all realizations, except for a set of measure ε.

For the linear problem this means constancy of the set of indices $i_1, \dots, i_n$ of the extremal vertex. The definition remains convenient also for cases of nonuniqueness of the solution of the linear problem, when the number of marked indices is less than n.

Here we restrict ourselves to presenting a stability criterion for the case of uniqueness of the solution of the linear problem. A point x is optimal if and only if the objective vector (in the present case the antigradient $F(x)$) can be represented as a linear combination, with nonnegative coefficients, of the outward normals to the boundary hyperplanes of the set of feasible plans passing through the point x (1).

Suppose that for some realization ($'$) of the constraints the extremum lies at the vertex $i_1', \dots, i_n'$, and for another realization ($''$) at the vertex $i_1'', \dots, i_n''$. Since the directions of the gradient of the objective function and of the normals do not change, the intersection of the hyperplanes $i_1', \dots, i_n'$ ceased to be a feasible plan under the realization ($''$), while the point of intersection of the hyperplanes $i_1'', \dots, i_n''$, on the contrary, became a feasible plan. Consequently, the point of intersection of the constraints $i_1', \dots, i_n'$ fell into the region

$$\sum_j a_{kj} x_j > b_k(\xi)$$

for some $k \neq i_1', \dots, i_n'$. Let us begin to move the constraints continuously from position ($'$) to position ($''$). In doing so, the point $i_1', \dots, i_n'$ with respect to the k-th constraint will pass from one half-space into the other. Consequently, at some moment, with an admissible position of all m constraints, the point $i_1', \dots, i_n'$ lies on the k-th constraint. From this one easily sees a sufficient condition for stochastic stability of the solution of the problem: the ellipsoid of dispersion of the point of the extremum must not intersect the hyperplanes of the lower constraints, except for the marked ones.

Suppose that, by methods of linear programming, the problem on average has been solved. Let us find the extremum on average x_0. Substitute the coordinates of the point x_0 into the equations of the lower hyperplanes for all indices except the marked ones; denote by d the min of the distances obtained; denote by $\max \sigma_i$ $(k = 1, \dots, m)$ the maximum of the standard deviations for the marked indices by σ. If the condition $d > q\sigma$ is satisfied, the stochastic linear programming problem is stable.

In what follows we shall assume the problem to be stable. Having solved the problem on average by the usual methods of linear programming, we find the extremum on average x_0. We single out the set of hyperplanes on which x_0 lies and fix their indices (marked indices). From the matrix of coefficients a_{ij} we single out the square submatrix A according to the marked indices. -

By known computational methods we find the inverse matrix A^{-1}. For each realization of the right-hand sides of the constraints—the column $B(\xi)$—the extremum point $\mathbf{x}_0(\xi)$ is obtained by the linear transformation

$$\mathbf{x}_0(\xi) = A^{-1} B(\xi). \tag{1}$$

Determining the vector $\mathbf{x}_0(\xi)$ for each particular realization requires only addition and multiplication operations. Therefore the stochastic problem can be posed on a computer once: solving the problem on the average, studying stability, extracting the matrix A, and finding A^{-1}. For each particular realization of the constraints, the optimal solution can be found by direct computation using formula (1).

The true extremum $\mathbf{x}_0(\xi)$ is a random point, and the maximum value of the objective function $F[\mathbf{x}_0(\xi)]$ is a random variable. By virtue of the linearity of $F(\mathbf{x})$, $MF(\mathbf{x}) = F(M\mathbf{x})$; that is, the mathematical expectation of the optimal value of the objective function $F[\mathbf{x}_0(\xi)]$ is equal to the value of the objective function at the extremum point on the average. The variance of $F[\mathbf{x}_0(\xi)]$ can also be computed. Introduce a coordinate system with the same origin and with axes parallel to the gradients of the marked constraints. Suppose that in this coordinate system

$y_1, \dots, y_n$ the objective function is written in the form $\Phi(y_1, \dots, y_n) = \sum_j c_i y_i$. In the coordinate system $y_1, \dots, y_n$ the components of the vector $\mathbf{x}_0(\xi)$ are independent. Therefore

$$DF[\mathbf{x}_0(\xi)] = D\Phi[\mathbf{y}_0(\xi)] = \sum_i c_i^2 \sigma_i^2.$$

We are able to write down a confidence interval for $F[\mathbf{x}_0(\xi)]$ with any significance level. Let us formulate the condition imposed on the right-hand sides of the constraints if it is desired that the variance of the optimal value of the objective function be less than δ. If the distribution parameters $b_i(\xi)$ are such that the vector of variances $\vec{\sigma}^{\,2}$ belongs to the elliptic cylinder $\sum_j c_i^2 \sigma_i^2 \le \delta$, then $DF[\mathbf{x}_0(\xi)] \le \delta$.

Questions of stability of parametric and stochastic programming problems in various formulations have been considered by various authors (2–5). Problems of this kind find application in economics and automatic control. In particular, the determination of optimal production volumes in prospective planning, based on a statistical forecast of consumption volumes, as well as the determination of optimal extraction volumes of natural raw materials, relying on a probabilistic estimate of prospective reserves, lead to the problem formulated above.

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Note: Figure translations are in progress. See original paper for figures.