S. V. USPENSKII

1965

SovietRxiv

Abstract

Full Text

S. V. USPENSKII

ON THE BOUNDARY PROPERTIES OF A CLASS OF DIFFERENTIABLE FUNCTIONS IN SMOOTH DOMAINS

(Presented by Academician S. L. Sobolev on 11 III 1965)

Numerous works are known that are devoted to the so-called embedding theorems for various classes of functions (see, for example, the survey by S. M. Nikol' skii ([1])). In all these works the most general classes of functions possessing different differential properties with respect to different variables are considered either in a parallelepiped with faces parallel to the coordinate planes, or in domains with such a boundary that for each of its points one can construct an analogous parallelepiped, fixed for the entire boundary, lying wholly in the domain. At the same time S. M. Nikol' skii constructed a number of examples which show that in an arbitrary smooth domain the embedding theorems for classes of functions having different differential properties with respect to different variables are, generally speaking, false in the formulation that holds for the whole space. In the present work, for general classes of functions summable with a weight, a certain class of domains is singled out in which embedding theorems analogous to those for the whole space hold. A number of propositions are also proved which are new even in the case of the whole space.

Let us introduce definitions. Let R_n be n-dimensional Euclidean space, R_n^+ the half-space $\{x_n \geq 0,\ (x_1, \dots, x_{n-1}) \in R_{n-1}\}$, and $\square$ the parallelepiped $\{a_i \leq x_i \leq b_i,\ i = 1, 2, \dots, n-1\}$.

Let G be an n-dimensional domain in R_n with boundary ∂G belonging to the class C^l, i.e., ∂G admits a decomposition into a finite number of intersecting pieces $\sigma_1, \dots, \sigma_N$, each of which has the representation

$$x_j = \varphi_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n), \tag{1}$$

where the index j, generally speaking, is different for each piece σ_k. $(x_1, \dots, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \in \square_j$, $\varphi_j \in C^l$, $|\partial^s \varphi_j / \partial x_i^s| < M$, $i \neq j$, $i = 1, 2, \dots, \dots, n$, $s = 1, 2, \dots, l$.

Let $\rho = \rho(x, \partial G)$ be the distance from the point x to ∂G. We shall say that $f \in L_{p_k}^{l_k}(G)$ if f has Sobolev generalized derivatives with respect to x_k up to order l_k and

$$\|f\|_{L^{l_k}_{p_k,\alpha_k}} = \left(\int_G \rho^{\alpha_k} \left|\frac{\partial^{l_k} f}{\partial x_k^{l_k}}\right|^{p_k} dG\right)^{1/p_k} < \infty,$$

$f \in L^l_{p,\alpha}(G)$, $l = (l_1, \dots, l_n)$, $p = (p_1, \dots, p_n)$, $\alpha = (\alpha_1, \dots, \alpha_n)$, if $f \in L^{l_i}_{p_i,\alpha_i}$, $i = 1, 2, \dots, n$.

Put

$$\|f\|_{L^l_{p,\alpha}} = \sum_{k=1}^{n} \|f\|_{L^{l_k}_{p_k,\alpha_k}},$$

$f \in W^l_{p_0,p,\alpha}$, if $f \in L_{p_0}$ and $f \in L^l_{p,\alpha}$.

Put

$$\|f\|_{W^l_{p,\alpha}} = \|f\|_{L_{p_0}(G)} + \|f\|_{L^l_{p,\alpha}}.$$

We shall also consider the classes $B^r_{p_0,p}(G)$, introduced by O. V. Besov ([2]).

We shall define $f(x_1, \dots, x_n) \in B^r_{p_0,p}(\partial G)$ if on each piece σ_k admitting a representation of the form (1), the function

$$f = f(x_1, \dots, \varphi_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n-1}), x_{j+1}, \dots, x_{n-1}) \in B^r_{p_0,p}(\square_j).$$

Let a class of functions $W^l_{p_0,p,\alpha}(G)$ be given in the domain. Let

$$\lambda = \max[\, l_k - \alpha_k/p_k\,], \qquad 1 \le k \le n.$$

We say that the domain G is regular with respect to the class $W^l_{p_0,p,\alpha}$ if its boundary ∂G can be decomposed into a finite number of intersecting pieces, each of which has the representation (1), and moreover

$$l_j - \alpha_j/p_j = \lambda, \tag{2}$$

and $l \ge l_j$.

The main result of the present work is the assertion that in regular domains the functions possess the same boundary properties as in the case of the whole space. Theorems 1, 2 and 3 generalize the corresponding results of V. P. Il' in and V. A. Solonnikov ([3]).

Theorem 1 (continuation). Let $f \in B^r_{p_0,p}(R_{n-1})$, $p = (p_1, \dots, p_{n-1})$.

Then, if p_n, α_n, l_n satisfy the conditions:

1) $p_n \geq p_i,\ i = 1, 2, \ldots, n-1;$

$$2)\quad \varkappa = 1 - \sum_{i=1}^{n-1} \frac{1}{r_i}\left(\frac{1}{p_i} - \frac{1}{p_n}\right) \neq 0;$$

$$3)\quad \gamma_i = \frac{pl_n - \alpha_n - 1}{p_n \varkappa r_i}\left[1 - \sum_{j=1}^{n-1} \frac{1}{r_j}\left(\frac{1}{p_j} - \frac{1}{p_i}\right)\right] > 0,$$

there exists an infinitely differentiable function $\mathfrak{F} = \mathfrak{F}(x_1, \ldots, x_n)$ such that

$$\mathfrak{F}\big|_{R_{n-1}} = f,$$

$$\sum_{i=1}^{n-1}\left(\int_{R_n^+} x_n^{p_i\gamma_i(l_i - r_i)-1}\left|\frac{\partial^{l_i}\mathfrak{F}}{\partial x_i^{l_i}}\right|^{p_i} dR_n^+\right)^{1/p_i} +$$

$$+\left(\int_{R_n^+} x_n^{\alpha_n}\left|\frac{\partial^{l_n}\mathfrak{F}}{\partial x_n^{l_n}}\right|^{p_n} dR_n^+\right)^{1/p_n} \leq c\|f\|_{B_p^r(R_{n-1})},$$

where $l_i > r$, and the constant c does not depend on f.

Theorem 2. Let l_n be an integer, α_n real, and

1) $p(l_n - k) - \alpha_n - 1 > 0,\ \alpha_n > -1;$

2) $\gamma_i = (pl_n - \alpha_n - 1)/pr_i^0;$

3) $r_i^\nu = r_i^0[p(l_n - \nu) - \alpha_n - 1]/[pl_n - \alpha_n - 1]$, $\nu = 0, 1, \ldots, k$.

Then, whatever the functions $f_\nu \in B_{p_\nu}^{r^\nu}(R_{n-1})$, $p = (p, \ldots, p)$, $\nu = 0, 1, \ldots, k$, there exists an infinitely differentiable function $\mathfrak{F}$, defined on R_n^+, such that

$$\frac{\partial^\nu \mathfrak{F}}{\partial x_n^\nu}\bigg|_{R_{n-1}} = f_\nu, \qquad \nu = 0, 1, \ldots, n;$$

$$\sum_{i=1}^{n-1}\int_{R_n^+} x_n^{p\gamma_i(l_i - r_i)-1}|D^l\mathfrak{F}|^p\, dR_n^+ + \int_{R_n^+} x_n^{\alpha_n}|D^{l_n}\mathfrak{F}|^p\, dR_n^+ \leq c\sum_{\nu=0}^{k}\|f_\nu\|_{B_p^{r^\nu}(R^{n-1})}^p.$$

Theorem 3. Let $\mathfrak{F} \in W_{p,\alpha}^l(R_n^+)$, $p = (p, \ldots, p)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $l = (l_1, \ldots, l_n)$;

1) $\alpha_i > -1 - p\nu,\ i = 1, 2, \ldots, n-1;$

2) $p(l_n - \nu) - \alpha_n - 1 > 0,\ 0 \leq \nu < l.$

Then

$$\partial^\nu \mathfrak{F}/\partial x_n^\nu\big|_{R^{n-1}} = f_\nu \in B^r_{p,p}(R_{n-1}),$$

where

$$r_i = [l_i(p(l_n-\nu)-\alpha_n-1)]/[pl_n-\alpha_n+\alpha_i], \tag{3}$$

$$\|f_\nu\|_{B^r_{p,p}} \le c\|\mathfrak{F}\|_{W^l_{p,\alpha}(R_n^+)}.$$

Theorems 2 and 3 carry over completely to regular domains. We formulate them for the case of domains of the form

$$\varphi(x_1,\dots,x_{n-1}) \le x_n \le \infty \qquad (x_1,\dots,x_{n-1}) \in R_{n-1},$$

where φ satisfies (2). The general case either reduces to this one or is obtained analogously. Put $\rho(x,\partial G) \le c|x_n-\varphi| \le c_1\rho(x,\partial G)$, $-\alpha_i/\rho\} = l_n-\alpha_n/p,\ 1 \le i \le n$. Then

Theorem 4. Let $\mathfrak{F} \in W^l_{p,\alpha}(G)$, $\alpha_i > -p\nu-1$, $\nu \ge 0$ an integer, $p(l_n-\nu)-\alpha_n-1 > 0,\ i=1,2,\dots,n$. Suppose also $\lambda = \max\{l_i-\alpha_i/\rho\} = l_n-\alpha_n/p,\ 1 \le i \le n$. Then

$$\partial^\nu \mathfrak{F}/\partial x_n^\nu\big|_{\partial G} = f_\nu \in B^r_{p,p}(\partial G),$$

where r_i satisfy (3),

$$\|f_\nu\|_{B^r_{p,p}(\partial G)} \le c\|\mathfrak{F}\|_{W^l_{p,\alpha}(G)}.$$

Theorem 5. Suppose the conditions 1), 2), 3) of Theorem 2 hold and, in addition,

$$0 < r_i^0 \le l_n-\alpha_n/p-1/p, \qquad i=1,2,\dots,n-1.$$

Then, whatever the functions $f_\nu \in B_p^{r_\nu}(\partial G)$ may be, there exists an l-times differentiable function, defined on G and satisfying the conditions

$$\partial^\nu \mathfrak{F}/\partial x_n^\nu\big|_{\partial G} = f_\nu,$$

$$\int_G |\mathfrak{F}|^p\,dG + \sum_{i=1}^n \int_G \rho^{\alpha_i}\left|\frac{\partial^{l_i}\mathfrak{F}}{\partial x_i^{l_i}}\right|^p dG \le c\sum_{i=0}^k \|f_\nu\|^p_{B_p^{r_\nu}(\partial G)},$$

where $\alpha_i = (pl_n - \alpha_n - 1)(l_i - r_i^0)/r_i^0 - 1,\ i = 1, 2, \dots, n-1$.

The following theorem on estimating mixed derivatives generalizes the corresponding results of S. M. Nikol' skii ([4]) and Yu. K. Solntsev ([5]).

Theorem 6. Let G be a bounded domain with smooth boundary $\partial G \in C^r$, $f \in W^r_{p,0}(G)$, $r = (2l, \dots, 2l)$, $p = (p, \dots, p)$, $1 < p < \infty$. Then f has all mixed derivatives and

$$\sum_{\sum \alpha_i = r} \int_G \left| \frac{\partial^r f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right|^p dG \le c \|f\|_{W^r_{p,0}(G)}.$$

Institute of Mathematics, Siberian Branch
Academy of Sciences of the USSR

Received
19 II 1965

REFERENCES

1. S. M. Nikol' skii, UMN, 16, 5 (101), 63 (1961).
2. O. V. Besov, Tr. Mat. Inst. im. V. A. Steklova AN SSSR, 61, 42 (1961).
3. V. P. Il' in, V. A. Solonnikov, DAN, 136, No. 3, 538 (1961).
4. S. M. Nikol' skii, Tr. Mat. Inst. im. V. A. Steklova AN SSSR, 61, 147 (1961).
5. Yu. K. Solntsev, Tr. Mat. Inst. im. V. A. Steklova AN SSSR, 63, 211 (1963).

Note: Figure translations are in progress. See original paper for figures.